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Mathematical Reviews

Vol. 20, No. 7 JULY-AUGUST, 1959 Reviews 4475-5117

LOGIC AND FOUNDATIONS

See also 4555.

4475:

McConnell, James. Whittaker's correlation of physics and philosophy. Proc. Edinburgh Math. Soc. 11 (1958), 57–68.

In this clearly written concise survey of Sir Edmund Whittaker's writings on the philosophy of physics, the author divides Whittaker's work in this field for convenience into five divisions: neo-Cartesianism, Eddington's principle, determinism and freewill, cosmology, natural theology. With regard to Eddington's Fundamental Theory, the author makes the point that in recent years Eddington's work would have been ignored had not Whittaker focussed attention on it, and he claims that both for this and for pointing out the epistemological defects he has placed both philosophers and mathematical physicists under a debt. *G. J. Whitrow* (London)

4476:

Hadamard, J. History of science and psychology of invention. Mat. Lapok 9 (1958), 64–66. (Hungarian)

The English original appeared in Mathematika 1 (1954), 1–3 [MR 16, 1].

4477:

Ackermann, Wilhelm. Ein typenfreies System der Logik mit ausreichender mathematischer Anwendungsfähigkeit. I. Arch. Math. Logik Grundlagenforsch. 4 (1958), 3-26.

Developing ideas presented in earlier papers [J. Symbolic Logic 15 (1950), 33–57; Math. Z. 55 (1952), 364–384; 57 (1953), 155–166; MR 12, 384; 14, 344, 834], the author describes a formal system which is not meant to be deductively complete (in the sense that a deduction theorem is valid for it) and whose combinatory completeness therefore does not necessarily entail inconsistency. It differs from earlier constructions of a similar kind in that stress is laid on mathematical adequacy rather than on provable consistency. Applications will follow later on. *E. W. Beth* (Amsterdam)

4478:

Härtig, Klaus. Ein Spezialfall der Substitution als Grundbeziehung der elementaren Semiotik. Z. Math. Logik Grundlagen Math. 3 (1957), 151–156.

This paper is a continuation of the author's article in the same Z. 2 (1956), 177–203 [MR 19, 933]. The relation Str $Z_1Z_0Z_2$ is defined to be Sub $Z_1Z_0?Z_2$, i.e., Z_2 is obtained from Z_1 by everywhere deleting Z_0. It is shown that an explicit elementary definition of Vk can be given in terms of Str. Thus, in view of the author's earlier paper, Str may be taken as the fundamental relation of semiotic. As in the earlier article, arguments are given in full detail. *W. W. Boone* (Urbana, Ill.)

4479:

Słomiński, J. Theory of models with infinitary operations and relations. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. 6 (1958), 449–456.

Let β be an ordinal and $\Delta=\{\alpha_\sigma\}_{\sigma<\beta}$ a β-sequence of ordinals. The rank ρ of Δ is the smallest initial ordinal such that $\bar{\alpha}_\sigma\leqq\bar{\rho}$. The dimension of Δ is the smallest initial $\gamma>\alpha_\sigma$ $(\sigma<\beta)$ such that γ is not cofinal with any $\alpha\leqq\alpha_\sigma$ for $\sigma<\beta$. A cardinal m is Δ-regular provided for every $\sigma<\beta$ and every set Γ_σ of cardinality $\leqq\bar{\alpha}_\sigma$ one has $\sum_k m_k<m$ $(k\in\Gamma_\sigma)$, m_k denoting a cardinal $<m$. A Δ-algebra is any sequence $\mathbf{A}=(A, F_0, F_1, \cdots, F_\sigma, \cdots)_{\sigma<\beta}$, where A is a non-empty set and F_σ is a relation of type α_σ in A. Further, absolutely free Δ-algebras (freely generated by a subset A_0 of A) are defined; the elements of A and A_0 are called terms and variables, respectively. They are used to define general models and logics $P_{\eta\tau}$ corresponding to such models, as well as the algebra

$$\mathbf{P}_{\eta\tau}=(P_{\eta\tau}, \rightarrow, ', \underbrace{\vee, \wedge}_{\omega_\eta}),$$

which is absolutely free of the type $(2, 1, 2, 2, \cdots, \xi, \xi, \cdots)$ freely generated by a set U_τ of atomic formulae. Other definitions are given (in particular with respect to the validity of formulae in $P_{\eta\tau}$) and various propositions are announced without proofs.

In particular, the algebra $(P_{\eta\tau}, \rightarrow, ')$ of the type $(2, 1)$ and the subalgebra $\mathbf{O}=(O, \rightarrow, ')$ generated by U_τ are discussed, and several "operations of consequence" as operations on subsets of O are defined.

Đ. Kurepa (Zagreb)

4480:

Mihailescu, Eugen. Formes normales dans l'ensemble $S(D)$ du calcul bivalent des propositions. An. Şti. Univ. "Al. I. Cuza" Iaşi. Secţ. I. (N.S.) 2 (1956), 15-27. (Romanian. Russian and French summaries)

On the basis of a formulation of the two-valued propositional calculus containing the functors C (implication), E (equivalence), A (alternation), and R ("reciprocity", i.e., non-equivalence), the author considers the introduction of an additional functor D, such that Dpq is true just when p is true and q false. This D is characterized by the axiom (in Łukasiewicz notation) $EDpqRApqq$. The author states three theorems; these give normal forms not containing D explicitly, to one of which any formula of one of the forms $D^np_1p_2\cdots p_{n+1}$ and $Dp_1Dp_2\cdots Dp_{n-1}Dp_np_{n+1}$ is equivalent. The motivation and significance of these theorems are not clear from the summaries. *H. B. Curry* (University Park, Pa.)

4481:

Pugmire, J. M.; and Rose, A. Formulae corresponding to universal decision elements. Z. Math. Logik Grundlagen Math. 4 (1958), 1–9.

A universal decision element is a logical function of $m \geqq 4$ arguments which, by a presetting of the inputs, can represent any logical function of two arguments. Certain universal decision elements with $m=4$ have already been discussed by A. Rose [C. R. Acad. Sci. Paris **244** (1957), 2343–2345; MR **19**, 239]. In the present paper a mechanical way of determining all possible universal decision elements of m arguments is discussed.

V. E. Beneš (Murray Hill, N.J.)

4482:

Cohen, L. Jonathan. Can the logic of indirect discourse be formalised? J. Symb. Logic **22** (1957), 225–232.

A relatively simple example is given of a statement presenting indirect discourse and resisting formalisation, in accordance with proposals for analysing indirect discourse made by Church and Carnap, because its components cannot be submitted to the hierarchy-principles which are usually adopted as a guarantee against semantical antinomies. Apparently the difficulty has been overlooked because of the preference which has been given to the Eubulides version of the Liar paradox, as compared to the Epimenides version. The Liar indicates a problem not merely about what we should take to be proper techniques of formalisation but also about what we should take to be its proper objects.

E. W. Beth (Amsterdam)

4483:

Kreisel, G.; and Wang, H. Applications of formalized consitsency proofs. II. Fund. Math. **45** (1958), 334–335.

Als Zusatz zu ihrer Arbeit gleichen Titels [Fund. Math. **42** (1955), 101–110; MR **17**, 447] skizzieren die Verff. wie Wf. Beweise von mengentheoretischen Axiomensystemen mit Klassenvariablen relativ zu solchen ohne Klassenvariable geführt werden können. Es wird dazu das interne Modell der Gödelschen "constructible sets" benützt, so daß die Axiome (z.B. das Ersetzungsaxiom) ohne gebundene Klassenvariable geschrieben werden können, da die Auswahlfunktion, und daher auch andere Funktionen, explizit definiert werden können. *P. Lorenzen* (Kiel)

4484:

Hajnal, András; und Kalmár, László. Eine Bemerkung zum Gödelschen Axiomensystem der Mengenlehre. I, II. Mat. Lapok 7 (1956), 26–42, 218–229. (Hungarian. Russian and German summaries)

An English version [Publ. Math. Debrecen **4** (1956), 431–449] was reviewed in MR **18**, 269.

4485:

Stupina, I. D. On some properties of the A_2-operation. Izv. Akad. Nauk SSSR. Ser. Mat. **21** (1957), 579-594. (Russian)

Aus Borelschen oder B-Mengen bekommt man durch Projizieren die A-Mengen oder A_1-Mengen; die Komplementen der A_1-Mengen heissen die CA- (oder CA_1-) Mengen; die Projektionen derselben heissen die A_2-Mengen; weiterhin baut man ähnlicherweise die Mengensysteme CA_2, A_3, CA_3 usw. Alle diese Klassen der sog. projektiven Mengen kann man durch andere Operationen erbauen. So z.B. wurden die A-Mengen zuerst durch A-Operation definiert; ähnlicherweise ordnet man jedem Doppelkomplex $\begin{matrix} a=m_1, & m_2, & \cdots, & m_t \\ b=n_1, & n_2, & \cdots, & n_k \end{matrix}$ der natürlichen Zahlen eine Menge $E_b{}^a$, so wird die A_2-Operation bzw. CA_2-Operation in bezug auf das Mengensystem $\{E_b{}^a\}$ als

$$A_2\{E_{n_1\cdots n_k}^{m_1\cdots m_t}\}=\bigcup_{m_1m_2\cdots}\bigcap_{n_1n_2\cdots}\bigcup_{k,t}E_{n_1\cdots n_k}^{m_1\cdots m_t}$$

bzw. als

$$CA_2\{E_b{}^a\}=$$
$$C(A_2\{CE_b{}^a\})=\bigcap_{m_1m_2\cdots}\bigcup_{n_1n_2\cdots}\bigcap_{k,t}E_b^a=\bigcap_{m_1m_2\cdots}\phi_{N'}\{\bigcap_t E_{\nu(n_1\cdots n_k)}^{m_1\cdots m_t}\}$$

erklärt; ν bzw. σ bedeute eine eineindeutige Abbildung der Menge aller endlichen Komplexe bzw. aller geordneter Paare von natürlichen Zahlen auf die Menge der natürlichen Zahlen. N' bzw. N'' ist eine starre Base der A-Operation bzw. der A_2-Operation. N'' besteht aus allen $(p_1p_2,\cdots)$, zu denen die Folgen $(m_1, m_2, \cdots)$, $(\nu_1, \nu_2, \cdots)$ $(t_1, t_2, \cdots)$ entsprechen so, dass $(\nu_1, \nu_2, \cdots)\in N_C'$, $p_i=\sigma(\nu_i;\ m_1, \cdots, m_{t_i})$ ist. Sei J der Bairesche Raum und $N\subseteq J$; dann hat man die bezügliche δs-Operation ϕ_N [s. Kozlova, Izv. **21** (1957), 349–370; MR **19**, 829; dieselbe Terminologie und Bezeichnungen werden gebraucht]. Die A_n-Operationen sowohl wie die CA_n-Operationen können durch starre Basen definiert werden; sei $N^{(n)}$ bzw. $N_C{}^{(n)}$ eine starre Base der A_n-Operation bzw. der CA_n-Operation. Für eine Basis N und eine Ordinalzahl $\alpha<\omega_1$ sei $\phi_{\check{N}_\alpha''}$ bzw. $\phi_{\check{N}_{*\alpha}}$ diejenige δs-Operation, die nur diejenige Punkte auswählt, welche durch Ketten einer starren reduzierten Basis der A_2-Operation definiert sind, deren Menge a) keine zerstreute Menge von Index $\leqq\alpha$ bilden bzw. b) sich als Vereinigung eines zerstreutes Mengensystem von Index $\leqq\alpha$ mit kompakter Hülle darstellen lässt. Sei E eine Menge; definieren wir $E^{[\alpha]}$, $E^{(\alpha)}$, $E^{(\alpha)}$ als E für $\alpha=0$ und als Menge, die man aus $E^{[\alpha-1]}$ bzw. $E^{(\alpha-1)}$ bzw. $E^{(\alpha-1)}$ erhält, nachdem man aus dieser Menge jeden isolierten Punkt bzw. jedes kompakte Stück bzw. jedes isolierte Stück, dessen Hülle kompakt ist, ausgesondert hat; falls α nichtisoliert ist, werden die bezüglichen Mengen als Durchschnitt der entsprechenden Mengen mit kleinerem "Exponent" definiert. Insbesondere werden die kleinsten Indizes β — Zerstreuungsindizes — betrachtet so, dass $E^{[\beta]}=0$ bzw. $E^{(\beta)}=0$ bzw. $E^{(\beta)}=0$ ist. Es werden dann u.a. die folgenden zwei Sätze bewiesen. Ist ein Mengensystem S invariant in bezug auf eine δs-Operation ϕ_N, deren Basis N eine A_2-Menge ist, so hat man $\phi_{\check{N}_\alpha''}(S)\subseteq S$ bzw. $\phi_{\check{N}_*{}^{(\alpha)}}(S)\subseteq S$; ist S^* so, dass $\phi_{N''}(S^*)$ der Klasse Q der A_2-Mengen angehört, so hat man $\phi_{\check{N}_\alpha''}(S)\subseteq Q$ bzw. $\phi_{\check{N}_*{}''^{(\alpha)}}(S^*)\subseteq Q$ (Th. 1 bzw. Th. 2). Für jede Klasse S der projektiven A_n-Mengen mit $n\geqq 2$ hat man $\phi_{\check{N}_*{}^{(\alpha)}}(S)\subseteq S$. *Đ. Kurepa* (Zagreb)

4486:

Stupina, I. D. On some properties of CA_2-operations. Izv. Akad. Nauk SSSR. Ser. Mat. **21** (1957), 835-862. (Russian)

Die Arbeit schliesst sich der vorangehenden eng an. Ist $n\geqq 2$, so ist $\check{N}^{(n)}$ bzw. $\check{N}_C{}^{(n)}$ nirgends dicht im $\check{J}$ (Lemma 1). N'' bzw. N_C'' steht in vollständig regulärer Beziehung zu jedem Mengensystem (Lemma 2; der Beweis enthält mehr als 10 Seiten). Für jedes Mengensystem S hat man

$$\phi_{N_c''\cdots\cdot}(S)\subseteq\phi_{N_c{}^{\nu}}(S),\quad \phi_{N_c''\cdots\cdot}(S)\subseteq\phi_{N_c''}(S)$$

(Th. 1; Beweis pp. 852–861). Kombiniert man diesen Satz mit einem Novikovschen Satze, so erhält man folgendes Korollar (Kor. 3): Im bekannten Gödelschen Axiomensystem Σ über Mengen, führt folgende Aussage X_n zu keinem Widerspruch, falls Σ selbst widerspruchsfrei ist: Von einem Index n ab, zu jeder Folge E_k der

CA_n-Mengen, die so beschaffen sind, dass in bezug auf die Folge E_k jeder Punkt der Menge $\phi_{N_C''}\{E_k\}$ höchstens $N_C''-\aleph_0$-deutig ist, entspricht eine Folge H_k von B-Mengen mit der Eigenschaft, dass $H_k \supseteq E_k$ und dass jeder Punkt der Menge $\phi_{N_C''}\{H_k\}$ höchstens $N_C''-\aleph_0$-deutig ist. Die Aussage X_2 kann man beweisen (Kor. 2).

Đ. Kurepa (Zagreb)

4487:

Trahtenbrot, B. A. Signalizing functions and tabular operators. Penzen. Gos. Ped. Inst. V. G. Belin. Uč. Zap. 4 (1956), 75–87. (Russian)

This paper is concerned with numerical functions and is motivated by Post's paper of 1944 [Bull. Amer. Math. Soc. **50** (1944), 284–316; MR **6**, 29]. The author defines a tabular operator in the following way. Let $\Gamma(n)$ associate to each n, by way of Gödel numbering, a sequence μ_n of distinct numbers $m_1, \cdots, m_k$, (k depending on n) and a table τ_n of 2^k rows and $k+1$ columns such that the element in row s and column r for $r \leqq k$, viz. i_r^s, is 0 or 1, while the element in the column $k+1$ and row s, viz. j_s, is a natural number. Let $f(n)$ be a given predicate, and let $\phi(n)$ be defined as follows: let $\Gamma(n)$ determine the sequence μ_n and the table τ_n; let s be the row in τ_n such that $i_1^s, \cdots, i_k^s$ is the same as $f(m_1), \cdots, f(m_k)$; then $\phi(n)=j_s$. (If $\phi(n)$ is a predicate, this gives a reduction of the decision problem for ϕ to that for f; it is Post's reduction by truth tables.) A tabular operation is one which associates such a ϕ to a given predicate f; it is primitive or general according as $\Gamma(n)$ is primitive or general recursive. The author investigates the relation of such operations to primitive recursive operations $T(f_1, \cdots, f_r)$. The latter is a uniform algorithm for generating a function ϕ from the initial functions $f_1, f_2, \cdots$, by the usual processes of primitive recursion. In connection with such operations he defines the notion of signalizing function, or signalizer, and that of resolvent for T.

Given T and $f_1, \cdots, f_r$, a signalizer, ϕ^*, for ϕ is a function whose value for given argument(s) is greater than any number used in the calculation of ϕ for the same argument(s); this is defined by induction on the steps in T, and the author shows how one can determine effectively such a signalizer which is strongly monotone, i.e., strictly increasing in each argument and never zero. The resolvent of an operator T on one function f is a primitive recursive function $F(t, m)$ such that

$$F(t, n)=T(\lambda x\cdot\exp(t, x); n),$$

where 'λ' is used à la Church, and $\exp(t, x)$ is the exponent of the xth prime in the expansion of t in terms of its prime factors. If $\phi^*(n)$ is a signalizer for f, and $\phi=T(f)$, then it is shown that

$$\phi(n)=F[\bar{f}(\phi^*(n)), n],$$

where $\bar{f}(m)$ is $2^{f(0)}\cdot 3^{f(1)}\cdots p_m{}^{f(m)}$; so that a primitive recursive T is characterized by its resolvent. If T acts only on predicates, a signalizer ψ can be defined which does once and for all as ϕ^* for any predicate f. Using this idea, the author shows that every primitive recursive T can be given in the form of a primitive tabular operator in which μ_n is $\{0, 1, 2, \cdots, \psi(n)\}$; and, conversely, any primitive tabular operator can be exhibited (in terms of a fixed Gödel numbering) as a primitive recursive operator. To extend this result to the case of a general tabular operator the author introduces a notion of Post operator, viz. one obtained from a primitive recursive one by replacing some primitive recursive functorial parameters by general recursive ones; then a Post operator on a predicate f is a general tabular one, and conversely.

H. B. Curry (University Park, Pa.)

4488:

Friedberg, Richard. A criterion for completeness of degrees of unsolvability. J. Symb. Logic **22** (1957), 159–160.

The notion of complete degree of recursive unsolvability was introduced by Kleene and Post in Ann. of Math. (2) **59** (1954), 379–407 [MR **15**, 772], to which the reader is referred for background. Since the publication of this fundamental work, the distribution of the complete degrees in the semi-lattice of all the degrees has remained obscure.

Now Friedberg shows that the complete degrees are precisely those degrees $\geqq$ the completion $0'$ of degree 0 ("solvability"). This characterization answers affirmatively the pressing questions of whether there exist incomparable complete degrees and whether particular degrees such as $a^{(\omega)}$ are complete. Moreover, the general result from which it comes may provide a useful approach to problems arising anywhere in the theory of unsolvability.

Friedberg's precise result is that, for any degree a, there is a degree b such that $b'=b\cup 0'=a\cup 0'$. In fact, given a function of degree a, a function of degree b can be defined by a uniform procedure.

G. F. Rose (Santa Monica, Calif.)

4489:

Kalmár, L. Über arithmetische Funktionen von unendlich vielen Variablen, welche an jeder Stelle bloss von einer endlichen Anzahl von Variablen abhängig sind. Colloq. Math. **5** (1957), 1–5.

For functions of a fixed, finite number of non-negative integral variables, with non-negative integral values, the concept of effective computability is formalized as general recursiveness. The author of this paper is concerned with a formalization of the same concept for functions $f(x_1, x_2, \cdots)$ of denumerably many variables. He observes that a necessary, but not sufficient, condition that such a function be "effective" is that it should be "finite"; i.e., for each sequence $x_1, x_2, \cdots$, the value of the function depends on only finitely many x_i's.

He introduces a transfinite classification of finite functions: the 0th class consists of the constant functions; for any ordinal α, the αth class consists of the functions $f(x_1, x_2, \cdots)$ which are not in any lower class, but are such that all of the functions obtained by fixing the value of x_1 are in lower classes. (Then for each finite k, the kth class consists of the functions of k variables. The ωth class consists of the functions such that the number of variables which determine the value of the function depends only on x_1). The author establishes some basic properties of this classification, and raises the problem of finding a normal form for finite functions.

The need for an extension of the concept of recursiveness (and of Church's Thesis) becomes evident with the realization that such a useful device as a Gödel numbering of finite sequences of objects (integers) can be considered as a finite function of the ωth class, and that the "effectiveness" of a Gödel numbering is a meaningful and significant, although at present informal, concept.

H. G. Rice (Pittsburgh, Pa.)

4490:

Mrówka, S. Recursive families of sets. Fund. Math. **44** (1957), 186–191.

The author gives some conditions under which denumerable unions or intersections of recursive or r.e. sets

will again be recursive or r.e. He also gives some conditions on recursive functions $f(n, m)$ under which $\min_n f(n, m)$ and $\max_n f(n, m)$ will be recursive functions of m. The proofs are straightforward, and some of the results are well known. *H. G. Rice* (Pittsburgh, Pa.)

4491:

Fine, N. J.; and Harrop, R. Uniformization of linear arrays. J. Symb. Logic 22 (1957), 130–140.

The first author [same J. **19** (1954), 41–44; MR **15**, 593] has already proved a conjecture contained in N. Goodman's 'The structure of appearance' [Harvard Univ. Press, Cambridge, Mass., 1951]. The main result of the present paper is the existence of an effective method for embedding a weakly mapped array consistently in a uniform array. This carries with it the result that it is possible to obtain effectively a minimal uniform extension of the original array, that is, a uniform extension with minimum span for such an extension and with the minimum number of elements possible for a uniform extension with this span. By way of conclusion, the authors mention a few questions not yet answered by their work.

E. W. Beth (Amsterdam)

4492:

Wang, Hao. A variant to Turing's theory of computing machines. J. Assoc. Comput. Mach. 4 (1957), 61-92.

Wang gives a further development of the theory of Turing [Proc. London Math. Soc. (2) **42** (1937), 230–265] on computing machines. He specifies a machine capable of solving all computation problems, only four basic types of instruction being used for the programmes. The machine uses only one symbol for marking and only one type of transfer and dispenses with erasing. The only types of instruction are "shift left one space", "shift right one space", "mark a blank space" and "conditional transfer". The paper includes a discussion of the possibility of a rapprochement between the theoretical approach of Turing and the practical design of digital computers.

A. Rose (Nottingham)

4493:

Wang, Hao. Universal Turing machines: an exercise in coding. Z. Math. Logik Grundlagen Math. **3** (1957), 69–80.

This paper divides roughly into two sections: the first is a short discussion of machine-reproducing machines and program-reproducing programs; the second is concerned with the following. Let B be the class of "basic machines", as defined previously by the author. There is a class P of programs such that, for $p \in P$, any $b \in B$ when equipped with p (and then regarded as another machine b') will have the property that it effects the computation that c would effect for program q, for all $c \in B$ and for all $q \in P$. The author constructs one such program p.

R. M. Baer (Berkeley, Calif.)

4494:

Oberschelp, Walter. Varianten von Turingmaschinen. Arch. Math. Logik Grundlagenforsch. **4** (1958), 53–62.

The work of a Turing machine can be analyzed into the operation of five machines, which: 1) move the tape one square to the left; 2) move the tape one square to the right; 3) place a mark in the scanned square; 4) erase a mark from the scanned square; 5) examine the scanned square, and, depending on whether a mark is observed, perform one of two prescribed actions. Two natural ways are given for representing numbers on the tape, as well as two natural ways for prescribing the beginning and end of a calculation. Then, under all four possible definitions, the class of computable functions is the class of recursive functions. Wang has shown that, under one of these definitions, the erasing machines are dispensable. However, the author proves that, in the three other cases, elimination of the erasing machines makes it impossible to calculate certain primitive recursive functions, and he gives, in each case, a precise description of the class of computable functions. *E. Mendelson* (New York, N.Y.)

SET THEORY

See also 4485, 4486, 4499, 4589.

4495:

Sedmak, Viktor. Sur les partitions des ensembles. Glasnik Mat.-Fiz. Astr. Društvo Mat. Fiz. Hrvatske Ser. II. **12** (1957), 17–19. (Serbo-Croatian summary)

Let S be a set; the author denotes by KS the cardinal number of S. Assume $KS \geqq \aleph_\alpha$ and consider partitions of S into subsets $S = \bigcup_j X_{i,j}$, $1 \leqq i < \aleph_\alpha$, where j runs through a set of power $\geqq \aleph_\alpha$. In other words S is partitioned in $\aleph_\alpha$ ways in at least $\aleph_\alpha$ parts; de Groot raised the problem if to each i, $1 \leqq i < \aleph_\alpha$, one can find an $f(i)$ so that the complement in S of $\bigcup_{1 \leq i < \aleph_\alpha} X_{i,f(i)}$ should have power $\geqq \aleph_\alpha$. The author proves this for $\alpha = 0$. He further remarks that for $\alpha = 0$ one can also show that $K \bigcup_{1 \leq i < \aleph_0} X_{i,f(i)} \geqq \aleph_0$. [See also N. G. de Bruijn and P. Erdös, Wiskundige Opgaven **20** (1956), no. 2, 18–19.] *P. Erdös* (Haifa)

4496:

Kreĭnin, Ya. L. About a property of sets effectively different from all Φ-sets. Dokl. Akad. Nauk SSSR (N.S.) **118** (1958), 237-238. (Russian)

Let ϕ, $\bar{\phi}$ be δs-operations; $\bar{\phi}$ is said to contain ϕ if there exists a mapping τ of the set $I\omega$ of ordinals $<\omega$ into itself such that for every space S and every ω-sequence F_n of subsets of S it is possible to associate an ω-sequence F_n' of sets $\subseteq S$ satisfying $F_n' = F_{\tau(n)}$ and $\bar{\phi}\{F_n'\} = \phi\{F_n\}$. For instance, the A-operation contains the lim inf-operation as well as the lim sup-operation. For other notations and terminology, see another paper by the author [Mat. Sb. N.S. **38(80)** (1956), 129–148; MR **17**, 950]. If $\bar{\phi}$ contains ϕ and if T is a set in a metric space S effectively different from all the $\bar{\phi}$-sets of S, then T is effectively distinct from all the ϕ-sets of S (Th. 1). Let S be a metric space and ϕ a δs-operation containing lim inf and lim sup. If T_0 of R is effectively distinct from all the ϕ-sets of S, then there are absolute G_δ-sets $E_i \subseteq T_i$ ($i = 0, 1$; $T_1 = S \setminus T_0$) and discontinuous $D_i \subseteq E_i$ such that E_i, D_{1-i} ($i = 0, 1$) be non-separable by absolute F_σ-sets (Theorem 2).

Đ. Kurepa (Zagreb)

4497:

Kreĭnin, Ya. L. On perfect compact nuclei of sets, effectively different from all Φ-sets. Dokl. Akad. Nauk SSSR (N.S.) **118** (1958), 436–438. (Russian)

The author deals with the problem of the existence of a perfect nucleus in non-projective sets, restricting himself to sets that are effectively different from every ϕ-set (ϕ being a δs-operation [cf. Kreĭnin, Mat. Sb. N.S. **38(80)** (1956), 129–148; MR **17**, 950]). If ϕ has the property that for every positive integer n_0 the basis of ϕ contains a chain $\{n_1, n_2, \cdots\}$ such that $n_0 < n_1 < n_2 < \cdots$ and if the space considered contains a T_ϕ (i.e., a set that is effectively different from every ϕ-set), then for every ϕ-set M

satisfying $M \subset T$ the set $T \cap (R \setminus M)$ contains a discontinuum (Theorem 1). For some ϕ-operations and some spaces R a similar result holds for ϕ-sets $X \supseteq T$: the set $X \cap (R \setminus T)$ contains a discontinuum (Theorem 2).

A necessary and sufficient condition for a property is established in order that it belong to every set T_ϕ. Neither measurability nor the Baire property is such a condition. *Đ. Kurepa* (Zagreb)

4498:

Kurepa, G. Généralisation de l'opération de Suslin, de celle d'Alexandroff et de la formule de de Morgan. Bull. Internat. Acad. Yougoslave. Cl. Sci. Math. Phys. Tech. **5** (1955), 97–107.

Let S be a partially ordered set, $O(S)$ be the class of all maximal ordered subsets of S, and $O'(S)$ be the class of all those maximal subsets of S that have no comparable pair of distinct elements and contain at least one element of every set in $O(S)$. For every $x \in S$, let $f(x)$ be a set. Let $\mathscr{C}$ denote the complement of a set relative to $\bigcup_{x \in S} f(x)$. The author shows that

$$\mathscr{C} \bigcup_{M \in O(S)} \bigcap_{x \in M} f(x) \supseteq \bigcup_{A \in O'(S)} \bigcap_{t \in A} \mathscr{C} f(t),$$
$$\mathscr{C} \bigcap_{M \in O(S)} \bigcup_{x \in M} f(x) \subseteq \bigcap_{A \in O'(S)} \bigcup_{t \in A} \mathscr{C} f(t),$$

and that the equality signs hold in each of the following three cases: (I) S is ordered; (II) S has no comparable pair of distinct elements; (III) S, or the corresponding inversely partially ordered set, is such that every non-empty ordered subset has an initial element and, for every $x \in S$, the set of predecessors of x in S is ordered.

F. Bagemihl (Notre Dame, Ind.)

4499:

Kurepa, George. On regressing functions. Z. Math. Logik Grundlagen Math. **4** (1958), 148–156.

Let S be a set with $|S| = \aleph_\sigma$ and $cf(\sigma) > 0$. Suppose that for every $x \in S$, $\varphi(x)$ is an ordinal number, $f(x) \in S$, $\varphi(f(x)) < \varphi(x)$; and for every $\alpha \in \varphi(S)$, $|\varphi^{-1}(\alpha)| < \aleph_{cf(\sigma)}$. Then there exists a subset E of S, where $\varphi(E)$ is of type $\omega_{cf(\sigma)}$, such that if x and y belong to E, $\varphi(f(x)) < \varphi(y)$. If, in addition, $cf(\sigma) = \sigma$, then there exists an element $x_0 \in S$ such that $|f^{-1}(x_0)| = \aleph_\sigma$. Applications to certain partitions of sets, as well as to partially ordered sets in which every chain is well-ordered (in particular, to dyadic trees), are given. *F. Bagemihl* (Notre Dame, Ind.)

COMBINATORIAL ANALYSIS

See also 4519.

4500:

Riordan, John. The combinatorial significance of a theorem of Pólya. J. Soc. Indust. Appl. Math. **5** (1957), 225–237.

The paper contains a careful exposition and proof of the theorem of the title, six classes of examples of the application of the theorem to combinatorial problems, and a bibliography of 13 items. From this point on we shall quote the author's introduction (with appropriate alterations) as the best possible review of reasonable length.

"The theorem of Pólya in question is the Hauptsatz of a famous paper [Acta Math. **68** (1937), 145–254] devoted to the enumeration of trees, linear graphs and chemical structures. It has been used with great effect in the field of its origin, but seems to have little currency elsewhere, though Pólya himself was careful to present it with the requisite generality for other uses. Hence the opportunity seems to be present for further exposition fortified by different examples, and it is to this that the present paper is directed.

Stated briefly, the combinatorial significance of the theorem is that it brings the consideration of enumerations dependent upon given order conditions within the scope of that powerful combinatorial tool, the generating function. The understanding of the theorem then requires the understanding of the generating function, and the means by which order conditions are specified, what Pólya calls the cycle index; hence their explanation precedes the theorem and its applications."

R. H. Bruck (Madison, Wis.)

4501:

Fuchs, Ladislaus. Ein kombinatorisches Problem bezüglich abelscher Gruppen. Math. Nachr. **18** (1958), 292–297.

The reviewer has shown [Proc. Amer. Math. Soc. **3** (1952), 584–587; MR **15**, 350] that in a finite Abelian group G of order n, additively written, if $a_1, \cdots, a_n$ are the elements of G and $b_1, \cdots, b_n$ a permutation of these elements, the differences $c_i = a_i - b_i$, $i = 1, \cdots, n$ satisfy the condition $c_1 + \cdots + c_n = 0$ and, conversely, if elements $c_1, \cdots, c_n$ satisfy the condition $c_1 + c_2 \cdots + c_n = 0$, then a permutation of the elements of G exists with these differences. This paper solves the same problem for infinite Abelian groups. Let the given differences be c_j each with an appropriate multiplicity, and let d_j be the subset of the c's which occur infinitely often. Only two conditions are necessary: (1) If the number of distinct c's is finite, and D is the group generated by the differences $d_i - d_j$, then the sum of $c_i - d_i$ over the remaining c's lies in D; and (2) If G is infinite cyclic, generated by g, then the differences d_i cannot be merely two values mg, $(m+1)g$. *Marshall Hall, Jr.* (Columbus, Ohio)

4502:

Ford, L. R., Jr.; and Fulkerson, D. R. Network flow and systems of representatives. Canad. J. Math. **10** (1958), 78–84.

If $\mathscr{S} = \{S_1, \cdots, S_n\}$ is a family of subsets of some set A, a function ϕ from $\mathscr{S}$ into A is called a system of representatives for $\mathscr{S}$ provided $\phi(S_j) \in S_j$ for all j. If in addition for each a_i in A one requires that the number of elements in $\phi^{-1}(a_i)$ lie between preassigned bounds $\alpha_i < \beta_i$, then ϕ is called a system of restricted representatives, and if $\alpha_i = 0$ and $\beta_i = 1$, ϕ is called a system of distinct representatives. Using their network flow theory [same J. **8** (1956), 399–404; MR **18**, 56], the authors give simple necessary and sufficient conditions for the existence of a system of restricted representatives. They also solve the problem of determining when two families of subsets $\mathscr{S} = \{S_1, \cdots, S_n\}$ and $\mathscr{T} = \{T_1, \cdots, T_n\}$ possess a common system of restricted representatives.

D. Gale (Providence, R.I.)

4503:

Steinberg, Donald A. Combinatorial derivations of two identities. Math. Mag. **31** (1957/58), 207–209.

4504:

Máté, János. On a problem in the history of Chinese mathematics. Mat. Lapok **7** (1956), 112–113. (Hungarian. Russian and English summaries)

The author gives an elementary proof of the identity

$$\sum_{j=0}^{k}\binom{k}{j}^2\binom{n+2k-j}{2k}=\binom{n+k}{k}^2.$$

From the author's summary

4505:

Nanjundiah, T. S. Remark on a note of P. Turán. Amer. Math. Monthly 65 (1958), 354.

This note gives a generalization of a combinatorial identity first proved by P. Turán [Mat. Lapok 5 (1954), 1–6; MR 16, 13]. The proof here is elementary whereas Turán used properties of Legendre polynomials. The generalized identity is

$$\sum_{r=0}^{\min(m,n)}\binom{m}{r}\binom{n}{r}\binom{\mu+r}{m+n}=\binom{\mu}{m}\binom{\mu}{n},\quad m,\ n=0,1,2,\cdots.$$

I. A. Barnett (Dublin)

ORDER, LATTICES

See also 4498, 4499, 4819.

4506:

Murata, Kentaro. A theorem on residuated lattices. Proc. Japan Acad. 33 (1957), 639–641.

The terminology used in the following is that of Birkhoff's "Lattice theory" [Amer. Math. Soc. Colloq. Publ. vol. 25, New York, 1948; MR 10, 673]. Let L be a cl-semigroup with a maximally integral identity e. The author proves first: L has a mapping into itself $a\to a^{-1}$ with the properties: 1) $aa^{-1}a\leqq a$, 2) $axa\leqq a\Rightarrow x\leqq a^{-1}$ if and only if L forms a residuated lattice.

Now let L be a residuated lattice with a maximally integral identity e. Suppose that: 1) the restricted descending chain condition holds for integral elements in L; 2) any prime element is divisor-free; 3) any prime element contains an element c satisfying $(c^{-1})^{-1}=c$. Then L forms a commutative cl-group which is a direct product of infinite cyclic groups generated by prime elements. This is an improvement of a result previously found by Asano and the author [J. Inst. Polytech. Osaka Univ. Ser. A. Math., 4 (1953), 9–33; MR 15, 502].

Št. Schwarz (Bratislava)

4507:

Kolibiar, Milan. Charakterisierung der Verbände durch die Relation "zwischen". Z. Math. Logik Grundlagen Math. 4 (1958), 89–100.

A set K in which a ternary relation abc is defined is called an m-system with respect to this relation. Two sets of conditions on an m-system are given so that a (not necessarily modular) lattice may be defined in K, where axb is related to the lattice "betweenness" relation: (1) $(a\frown x)\smile(x\frown b)=x=(a\smile x)\frown(x\smile b)$. The first is a set of sufficient conditions so that axb implies (1); this is closely related to a set of necessary and sufficient conditions for the existence of a modular lattice by Kelly [Duke Math. J. 19 (1952), 661–669; MR 14, 494]. The second is a set of necessary and sufficient conditions in terms of closure and segment (defined as below) that assure the existence of a lattice in which axb is equivalent to (1). Let $B(a,b)=[x|axb]$. If $a,b\in A\Rightarrow B(a,b)\subseteq A$, A is closed. The closure $\bar{A}$ of a set A is the intersection of all closed sets $C\supseteq A$. $M(a,b)=\overline{B(a,b)}$; if $B(a,b)=M(a,b)$, it is denoted by (a,b) and is a segment. The conditions are: (A) any three elements of K are contained in a segment; (B) $M(a,b)\frown M(b,c)\frown M(c,a)\neq 0$; (C) $axb\Leftrightarrow M(a,x)\frown M(b,x)=\{x\}$; and (F) segments may be "oriented", i.e., there is a mapping which associates to each segment $J=(a,b)$ a pair of elements $o,u\in K$, so that $J=(o,u)$ and so that, if $(o,u)\subseteq(o',u')$ and the segment (o',u) exists, $o\in(o',u)$. If K is itself a segment (o,u), (F) is not needed, and the other three properties are independent. The author remarks that he has not succeeded in finding an example of an m-system in which (A), (B), and (C) hold, but (F) does not.

A. A. Grau (Oak Ridge, Tenn.)

4508:

Matsushima, Yataro. On the relations "semi-between" and "parallel" in lattices. Proc. Japan Acad. 34 (1958), 341–346.

In this paper the author extends his earlier work [same Proc. 32 (1956), 549–553; MR 18, 713]. He introduces the concepts "semi-betweenness" and "parallelism" in lattices. Modular and distributive lattices are characterised by properties of semi-betweenness. The notion of parallelism is applied in lattice-polygons, in particular, in lattice quadrangles.

Ph. Dwinger (Lafayette, Ind.)

4509:

Roy, Kamalaranjan. Dual Newman algebra. Bull. Calcutta Math. Soc. 49 (1957), 177–187.

A postulational study of "dual Newman algebras" — i.e., of Newman algebras with inclusion dualized. The discussion is essentially isomorphic to that of Newman algebras [G. D. Birkhoff and G. Birkhoff, Trans. Amer. Math. Soc. 60 (1946), 3–11; MR 8, 192].

G. Birkhoff (Cambridge, Mass.)

4510:

Bruns, Günter; und Schmidt, Jürgen. Ein Zerlegungssatz für gewisse Boolesche Verbände. Abh. Math. Sem. Univ. Hamburg 22 (1958), 191–200.

The authors discuss the decomposition of an ideal $\mathfrak{D}$ in the lattice of all subets of a given set E into the direct sum of two sublattices $\mathfrak{D}_H$ and $\mathfrak{D}_{E-H}$, the traces of $\mathfrak{D}$ in H and its complement, in such a way that $\mathfrak{D}_H$ is the lattice of all subsets of H, and $\mathfrak{D}_{E-H}$ contains only maximal elements.

M. M. Day (Urbana, Ill.)

4511:

Andreoli, Giulio. Struttura delle algebre di Boole e loro estensione quale calcolo delle classi (in senso ordinario oppure probabilistico). Giorn. Mat. Battaglini (5) 5(85) (1957), 141–171.

One considers the elementary Boolean algebras in connexion with the theory of probability. The values 1 and 0 are interpreted as a sure choice and a sure non-choice respectively; in general for every $p\in R[0,1]$ p is meant as a probability of a choice. Let U be a non-empty universe (a set) and $x\in U$. One considers mappings f of U into the set $R[0,1]$ of reals x, $0\leqq x\leqq 1$, and into $\{0,1\}$ in particular; if, for $x\in U$, $fx=p$, p is considered as the probability of the f-choice of the atom x. To any ordered pair f, g of such mappings one associates other mappings, in particular, fg and $f+g$ defined by the relations $(fg)x=\inf\{fx,gx\}$, $(f+g)x=\sup\{fx,gx\}$ for each $x\in U$. For the Boolean case that the values are 0, 1, the preceding Boolean definitions of fg, $f+g$ are equivalent to the arithmetical definitions $(fg)x=fxgx$, $(f+g)x=fx+gx-fxgx$. In the probabilistic case these arithmetical definitions are different from the Boolean definitions; e.g., one has neither tautology nor distributive laws; for instance, $[(f+g)h]x=(fx+gx-fxgx)hx\neq fxhx+gxhx-fxgx(hx)^2=(fh+gh)x$. The arithmetical product $(fg)x$ is

commutative and associative and is a bilinear function of fx, $1-fx$ and gx, $1-gx$; the converse is proved to hold also. The direct product of elementary Boolean algebras, as well as the Boolean algebras in connexion with a given probability model of algebra, is considered too. For instance, if f is any mapping of U into $R[0, 1]$, then for each $0\leqq p\leqq 1$ one could consider the set $\{f^{-1}p\}=\{x|x\in U, fx=p\}$, the corresponding partition $\mathscr{U}=\bigcup_p \{f^{-1}p\}$ of $\mathscr{U}$, and the Boolean function which is identical with the characteristic function of the set $\{f^{-1}p\}$.
Đ. Kurepa (Zagreb)

GENERAL ALGEBRAIC SYSTEMS

See also 4479, 4600.

4512:

Shoda, Kenjiro. Berichtigungen zu den Arbeiten über die Erweiterungen algebraischer Systeme. Osaka Math. J. **9** (1957), 239-240.

Correction of a theorem in Osaka Math. J. **4** (1952), 133–144 [MR **14**, 614]. *O. Ore* (New Haven, Conn.)

4513:

Boccioni, Domenico. Indipendenza delle condizioni di associatività negli ipergruppoidi. Rend. Sem. Mat. Univ. Padova **27** (1957), 228–244.

A hyper-groupoid (ipergruppoide) H^0 is a set H together with a mapping ϕ of H^2 into the set of all non-null sub-sets of H. The element (x, y, z) of H^3 is called associative in H^0 if $\phi(x, \phi(y, z))=\phi(\phi(x, y), z)$. This equation is called a condition of associativity. The conditions of associativity are said to be independent for H if there exists, for each element of H^3, a ϕ such that the element is the one and only element of H^3 which is not associative in H^0. The main theorem is that the conditions of associativity are independent for H if and only if the cardinal of H is greater than 2. *H. A. Thurston* (Vancouver, B.C.)

4514:

Boccioni, Domenico. Ipergruppoidi di ordine minimo in cui una data terna è isolata. Rend. Sem. Mat. Univ. Padova **27** (1957), 350–374.

This paper is suggested by the preceding one. An element of H^3 is said to be isolated in H^0 if it is the one and only element of H^3 which is not associative in H^0. The problem tackled here is to determine (to within an isomorphism) all hyper-groupoids of lowest order in which a given element is isolated. It follows from the previous paper that if the element (of H^3) is of the form (a, a, b), (a, b, a) or (b, a, a), the lowest order is 2 (this case is solved completely) and that if the element is of the form (a, a, a) or (a, b, c), the lowest order is 3 (this case is partly solved). *H. A. Thurston* (Vancouver, B.C.)

4515:

Boccioni, Domenico. Indipendenza delle condizioni di distributività. Rend. Sem. Mat. Univ. Padova **28** (1958), 1–30.

If B^0 is an algebra with two binary operations (written as addition and multiplication) and carrier B, then the element (x, y, z) of B^3 is called left-distributive in B^0 if

$$x(y+z)=(xy)+(xz).$$

The "conditions of left-distributivity" are said to be independent for B if there exists, for each element of B^3, an addition and a multiplication for which it is the one and only element of B^3 which is not distributive in B^0. The main theorem is that the conditions of distributivity are independent for B if and only if the cardinal of B is greater than 2. *H. A. Thurston* (Vancouver, B.C.)

4516:

Boccioni, Domenico. Indipendenza delle condizioni di mutua distributività. Rend. Sem. Mat. Univ. Padova **28** (1958), 40–49.

Statements of the following forms are called "conditions of mutual distributivity" in a set B:

$$x(y+z)=xy+xz,\ (x+y)z=xz+yz,$$
$$x+(yz)=(x+y)(x+z),\ (xy)+z=(x+z)(y+z),$$

where x, y, and z are elements of B.

If B is a set with cardinal greater than 3, and if a condition of mutual distributivity on B is given; then there exist an addition and a multiplication on B for which the given condition of mutual distributivity is the one and only condition of mutual distributivity on B which is not true. *H. A. Thurston* (Vancouver, B.C.)

4517:

Boccioni, Domenico. Dipendenza delle condizioni di mutua distributività nei bisistemi di ordine 3. Rend. Sem. Mat. Univ. Padova **28** (1958), 50–67.

The converse of the theorem in the preceding paper. *H. A. Thurston* (Vancouver, B.C.)

THEORY OF NUMBERS

See also 4548, 4593, 4663.

4518:

Sierpiński, W. Sur quelques problèmes concernant les points aux coordonnées entières. Enseignement Math. (2) **4** (1958), 25–31.

It is proved that there are circles and squares in the plane containing in their interior a given number of points with integral coordinates.
C. G. Lekkerkerker (Amsterdam)

4519:

Milošević, Kovina. Sur quelques sommes finies. Bull. Soc. Math. Phys. Serbie **8** (1956), 111–116. (Serbo-Croatian summary)

Dans cette note nous appliquons une méthode du calcul des différences finies pour effectuer la sommation de l'expression

$$\sum_{k=0}^{m}\left\{(-1)^k\binom{m}{k}\Big/\prod_{i=0}^{\nu-1}(n+kp+is)\right\},$$

où m, ν sont des nombres naturels et n, p, s désignent des nombres réels. Ces sommes ont fait également le sujet d'une note de D. S. Mitrinovitch [Univ. Beogradu. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. no. 7 (1956); MR 20 #1650]. *De l'introduction*

4520:

Robinson, Raphael M. The converse of Fermat's theorem. Amer. Math. Monthly **64** (1957), 703–710.

Discussion and proofs of known results concerning the use of the converse of Fermat's theorem as a test for primeness. Also some theorems in new forms,

e.g.: Let $N=kQ+1$, $0<k<Q$. If $a^{N-1}\equiv 1 \pmod N$ and $(a^{(N-1)/q}-1, N)=1$ for every prime factor q of Q, then N is prime. Let $N=k\cdot 2^n+1$, $n>2$, $0<k<6\cdot 2^n+7$. If now $(a/N)=-1$, then N is prime, if and only if $a^{\frac{1}{2}(N-1)}\equiv -1 \pmod N$. *N. G. W. H. Beeger* (Amsterdam)

4521:

Campbell, J. G. Diophantine problems having no solution. Amer. Math. Monthly **65** (1958), 204.

The author gives a criterion that the Diophantine equation

$$A_1x_1^{r_1(p-1)}+A_2x_2^{r_2(p-1)}+\cdots+A_nx_n^{r_n(p-1)}=B,$$

where p is a prime, has no solution. His result is an immediate consequence of Fermat's theorem and well known. *A. Brauer* (Chapel Hill, N.C.)

4522:

Schinzel, André. Sur l'existence d'un cercle passant par un nombre donné de points aux coordonnées entières. Enseignement Math. (2) **4** (1958), 71–72.

Using the fact that the diophantine equation $x^2+y^2=5^k$ (k a suitable positive integer) has $4(k+1)$ solutions, the author solves in the affirmative sense the problem posed in the title. *C. G. Lekkerkerker* (Amsterdam)

4523:

Moessner, Alfred. Folgerungen aus den Gleichungen $A\cdot B=C\cdot D$ und $a^2+b^2=c^2$. Bul. Inst. Politehn. Iaşi (N.S.) **3** (1957), 7–14. (Russian and Romanian summaries)

Various identities, involving equal sums of like powers, are established. *W. H. Mills* (New Haven, Conn.)

4524a:

Zeckendorf, E. La valeur nulle ou négative du discriminant dans les équations quadratiques. Bull. Soc. Roy. Sci. Liège **27** (1958), 68–73.

4524b:

Zeckendorf, E. Equations quadratiques à discriminant carré. Bull. Soc. Roy. Sci. Liège **27** (1958), 128–141.

These articles deal with the Lucas recurrent sequences $u_{n+2}=au_{n+1}+bu_n$ (a, b, u_0, u_1 rational integers) which admit solutions $D_n=\alpha^n+\beta^n$, $S_n=(\alpha^n-\beta^n)/(\alpha-\beta)$, α, β roots of $x^2-ax-b=0$, $n=0, 1, 2, \cdots$. The identity $(\alpha^n+\beta^n)^2-(\alpha^n-\beta^n)^2=4(\alpha\beta)^n$ leads to the diophantine equation (1) $x_n^2-\Delta y_n^2=4(-b)^n$, $\Delta=a^2+4b$, with the set $\{D_n, S_n\}$ as solutions.

The author refers to some of his own papers on this subject [same Bull. **25** (1956), 574–584; **26** (1957), 112–122; **27** (1958), 28–40; MR **19**, 523, 730; **20** #1660], but since no other author is mentioned it is not easy to determine the relation of the new results to known theorems.

We reproduce in essence his résumé of part I: "Quand on les considère comme des suites de récurrentes, les progressions arithmétiques ... et les progressions géométriques correspondent à la valeur nulle du discriminant. Pour une valeur négative du discriminant se trouve réalisée une progression géométrique à raison complexe et dont les termes doivent s'inscrire sur une spirale logarithmique..." The résumé of II contains frequent references to the text, and is omitted.

Both articles examine the solution of (1) in terms of Lucas-Fibonacci sequences. I is devoted to the case $\Delta=0$ and to some aspects of $\Delta<0$. II treats the case $|\Delta|=$the square of a rational integer. The papers contain special results in apparently definitive form; but the reader is perhaps not helped by the effort of the author to incorporate in his notation all parameters involved, so that, for example, equ. (1) appears in the form

$${}^{m,n}_{a,b}\tau_x^2-(n^2+m)\cdot{}^{m,n}_{a,b}t_x^2=4k_1\cdot(-m)^x.$$

A. J. Kempner (Boulder, Colo.)

4525:

Boico, I. Note sur le théorème de Fermat. Gaz. Mat. Fiz. Ser. A (N.S.) **10(63)** (1958), 609–612. (Romanian. French and Russian summaries)

Etant donné que l'équation $x^3+y^3=z^3$ n'a pas de solutions en nombres entiers, l'auteur donne quelques solutions de l'équation $x^3+y^3=z^3+1$ et montre qu'on peut obtenir des solutions en nombres entiers arbitrairement grands. *Résumé de l'auteur*

4526:

O'Meara, O. T. The integral representations of quadratic forms over local fields. Amer. J. Math. **80** (1958), 843–878.

The author finds necessary and sufficient conditions that will determine the integral representations of an arbitrary quadratic form over any local field in which 2 is either a unit or a prime. His criteria are in terms of the field representations of forms derived from certain canonical decompositions of the given forms. The exact criteria are stated in terms of the lattices associated with the forms, and the treatment is from this point of view. Since the notation is intricate, space does not permit giving the detailed conditions in this review.

B. W. Jones (Boulder, Colo.)

4527:

Min, Szu-Hoa. A generalization of the Riemann ζ-function. III. The mean-value theorems for $Z_{n,k}(s)$. Acta Math. Sinica **6** (1956), 347–362. (Chinese. English summary)

[For Parts I and II, see Acta Math. Sinica **5** (1955), 285–294; **6** (1956), 1–11; MR **17**, 462; **18**, 112]. Let $\delta>k\nu$ and

$$z_{n,k}(s)=\sum_{x_1=-\infty}^{+\infty}\cdots\sum_{x_k=-\infty}^{+\infty}{}'(x_1^n+\cdots+x_k^n)^{-s},$$

where the term with $x_1=\cdots=x_k=0$ is excluded; further $\nu=1/n$. The author continues his work of parts I and II and proves the following two mean value theorems. (A) Let $0<a<k\nu-\nu$ and let an not be an integer. Then, as $\delta\to 0+$,

$$\int_0^\infty t^{2a-1}|z_{n,k}(a+it)|^2e^{-2\delta t}dt$$
$$=c_1\delta^{-2(n-1)(k\nu-\nu-a)-1}(1+o(1))+O(\delta^{-2a})+O(\delta^{-4-\varepsilon}),$$

where $\varepsilon>0$ and

$$c_1=k^2(2\pi)^{\frac{1}{2}}(2\pi\nu)^{-2(k-na-1)-1}(2\Gamma(1+\nu))^{2k-2}$$
$$\times\Gamma(2(1-\nu)(k-na-1)+1)\zeta(2(k-na)).$$

(B) Let $an>0$ not be an integer, and let $2(n-1)(k\nu-\nu-a)+1>\max(2a, 4)$. Then, as $T\to+\infty$,

$$\int_0^T|z_{n,k}(a+it)|^2dt\sim c_3T^{2(n-1)(k\nu-\nu-a)-2a+2},$$

where

$$c_3=2^{-2(k\nu-\nu)+1}[2(n-1)(k\nu-\nu-a)-2a+2]^{-1}$$
$$\times\pi^{-2(k-na)+3/2}k^2\nu^{2na}\Gamma(\nu)^2\zeta(2(k-na)).$$

K. Mahler (Manchester)

4528:

van Lint, J. H. Über einige Dirichletsche Reihen. Nederl. Akad. Wetensch. Proc. Ser. A **61**=Indag. Math. **20** (1958), 56–60.

Sandham [Quart. J. Math. Oxford Ser. (2) **4** (1953), 230–236; MR **15**, 288] has given Euler product expansions for the Dirichlet series $\sum_m r_a(m^2)m^{-s}$, where $r_a(m^2)$ is the number of representations of m^2 as a sum of a squares, for $a=3, 5, 7$. For a odd >8, such simple formulas do not exist, since the genus of the quadratic form $\sum_{i=1}^a x_i^2$ contains more than one class. The author replaces $r_a(m^2)$ by $R_a(m^2)$, a weighted mean of the representations of m^2 in the genus of the form, and sums over odd m only. The resulting Dirichlet series then has a very simple form, of which Sandham's formulas are special cases. It is stated that a generalization to determinant $D>1$ can be given, in which the summation is over those m prime to $2D$.

N. J. Fine (Princeton, N.J.)

4529:

Kasch, Friedrich. Wesentliche Komponenten bei Gitterpunktmengen. II. J. Reine Angew. Math. **199** (1958), 53–55.

The author gives a sharpened version of a theorem proved by him in an earlier paper of the same title [same J. **197** (1957), 208–215; MR **19**, 14; theorem 2]. He mentions that his original proof contained some misprints and slips, and he therefore describes his new argument in detail. (See the review of the earlier paper for notation.) The theorem now reads: If A is a set of r-dimensional non-negative lattice points with density α, $0<\alpha<1$, if B is any basis containing 0 with average order $\lambda<\infty$, and if γ is the density of the set $A+B$, then γ satisfies

$$\gamma\geq\alpha\Big(1+c\frac{1-\alpha}{\lambda}\Big),$$

where c may take any value which, for arbitrary $q>1$, is given by

$$c=c(q)=\frac{1}{(r+1)^{r+1}q^r}\min_{k=1,\cdots,r}\{(q-1)^k(r+1-k)\}.$$

Taking $q=3$, not necessarily the best choice, already gives a better result than the earlier version.

H. Halberstam (London)

4530:

Wang, Yuan. On the representation of large even integer as a sum of a prime and a product of at most 4 primes. Acta Math. Sinica **6** (1956), 565–582. (Chinese. English summary)

Assume the truth of the Riemann hypothesis in the stronger form that the real parts of all zeros of all Dirichlet L-series are $\leq\frac{1}{2}$. Then the author can prove the following theorems. (1) Every large even integer is the sum of a prime and of a product of at most 4 primes. (2) There are infinitely many primes p such that $p+2$ is a product of at most 4 primes. (3) If $\varepsilon>0$, then the number of prime pairs p, $p+2$ not exceeding N is not greater than

$$(8+\varepsilon)\prod_{p>2}\Big(1-\frac{1}{(p-1)^2}\Big)\frac{N}{(\log N)^2}+O\Big(\frac{N}{(\log N)^3}\Big).$$

K. Mahler (Manchester)

4531:

Wang, Yuan. On some properties of integral valued polynomials. Advancement in Math. **3** (1957), 416–423. (Chinese. English summary)

Let $F(x)$ be an irreducible integral-valued polynomial of degree k without fixed prime divisors, and let $\pi(N; F(x))$ denote the numbers of primes represented by $F(x)$ for $x=1, 2, \cdots, N$. Let $h_m(k!m)$ denote the number of solutions of $F(x)\equiv0 \pmod m$ with $0\leq x<k!m$, and let $\omega(m)=h_m(k!m)/k!$. Then, by means of Selberg's method, the author proves that

$$\pi(N; F(x))\leq2\prod_p\Big(\frac{1-\omega(p)/p}{1-1/p}\Big)\cdot\frac{N}{\log N}+o\Big(\frac{N}{\log N}\Big).$$

He also considers similar problems for reducible polynomials and gives, e.g., an upper bound for the number of x's for which both x^2+1 and x^2+3 are primes.

K. Mahler (Manchester)

4532:

Chen, Jun-jing. On Waring's problem for n-th powers. Acta Math. Sinica **8** (1958), 253–257. (Chinese. English summary)

The author proves that the number $G(n)$ in Waring's problem is not greater than $n(3 \log n+5.2)$. This slightly improves Vinogradov's estimate with 11 instead of 5.2.

K. Mahler (Manchester)

4533:

Wang, Yuan. A note on some properties of the arithmetical functions $\varphi(n)$, $\sigma(n)$ and $d(n)$. Acta Math. Sinica **8** (1958), 1–11. (Chinese. English summary)

Let $\varphi(n)$, $\sigma(n)$, and $d(n)$ be Euler's function, the sum, and the number, of the divisors of n. The following results are proved. (1) If $\varepsilon>0$, and if $a_1, \cdots, a_k$ are finitely many non-negative numbers, there exist positive constants $c_0(a, \varepsilon)$ and $X_0(a, \varepsilon)$ such that, for $x>X_0$, the interval $1<p\leq x$ contains more than

$$\frac{c_0x}{(\log x)^{k+2}\log\log x}$$

primes p satisfying

$$\left|\frac{\varphi(p+\nu+1)}{\varphi(p+\nu)}-a_\nu\right|<\varepsilon \quad (\nu=1, 2, \cdots, k).$$

(2) A result of the same form holds if φ is replaced by σ. (3) For every positive integer k there exists a constant $\gamma>1$ as follows. If $a_0, a_1, \cdots, a_k$ are any $k+1$ positive integers, there are two positive constants $c_2(a)$ and $X_2(a)$ such that, if $x>X_2$, the interval $1<p\leq x$ contains more than

$$\frac{c_2x}{(\log x)^{k+2}\log\log x}$$

primes p satisfying

$$a_\nu\leq d(p+\nu+1)\leq\gamma a_\nu \quad (\nu=0, 1, 2, \cdots, k).$$

(4) If each of $a_1, \cdots, a_k$ is either 0 or ∞, there exists an infinite sequence of primes $\{p_j\}$ satisfying

$$\lim_{j\to\infty}\frac{\varphi(p_j+\nu+1)}{\varphi(p_j+\nu)}=a_\nu \quad (\nu=1, 2, \cdots, k).$$

K. Mahler (Manchester)

4534:

Erdős, Pál. Remarks on two problems of the Matematikai Lapok. Mat. Lapok **7** (1956), 10–17. (Hungarian. Russian and English summaries)

"Let $L_n=[\log_3 n/\log_4 n]$ ($\log_r n$ denotes the r times iterated logarithm). Then for every ε and $n>n_0$ there exists an $a<n$ for which

$$\varphi(a)+\varphi(a+1)+\cdots+\varphi(a+L_n)<\varepsilon a.$$

On the other hand for every $\eta>0$

$$\varlimsup_{a\to\infty}\frac{\varphi(a)+\varphi(a+1)+\cdots+\varphi(a+(1+\eta)L_n)}{a}=\infty.$$

Further, it is stated without giving the proof that

$$\varlimsup_{a\to\infty}\frac{1}{a}\left\{\varphi(a)+\cdots+\varphi\left(a+\frac{\log_3 a}{\log_4 a-\log_5 a}+\frac{c\log_3 a}{(\log_4 a)^2}\right)\right\}=\frac{e^c}{a},$$
$$\alpha=\prod_p (1-p^{-1})^{-1/p}.$$

Several other results are stated without proof; here we mention only one: Put $k_n=\log_3 n/\log_6 n$ and let $i_1, i_2, \cdots, i_{k_n}$ be any permutation of the integers $1, 2, \cdots, k_n$. Then for $n>n_0$ there exists an $a<n$ so that

$$\varphi(a+i_1)>\varphi(a+i_2)>\cdots>\varphi(a+i_{k_n}).$$

On the other hand, for $n>n_0(\varepsilon)$

$$\varphi(n)>\varphi(n+1)>\cdots>\varphi(n+(1+\varepsilon)k_n)$$

can not hold. The same results hold for $\sigma(n)$ instead of $\varphi(n)$." *Author's summary*

4535:

Rényi, Alfréd. Probability methods in number theory. Advancement in Math. 4 (1958), 465–510. (Chinese)

Expository paper. *K. L. Chung* (Syracuse, N.Y.)

4536:

Wegner, Kenneth W. Equations with trigonometric values as roots. Amer. Math. Monthly **66** (1959), 52–53.

The author states, "The purpose of this note is to make readily available for classroom use the 64 irreducible polynomial equations with integral coefficients and of degree two through seven whose roots are of the form $\pm\sin y$, $\pm\cos y$, $\pm\tan y$, $\pm\cot y$, $\pm\sec y$, or $\pm\csc y$, where y is a rational number of degrees."

4537:

✱Kleboth, Heinrich. Untersuchung über Klassenzahl und Reziprozitätsgesetz im Körper der 6l-ten Einheitswurzeln und die Diophantische Gleichung $X^{2l}+3^lY^{2l}=Z^{2l}$ für eine Primzahl l grösser als 3. Inaugural-Diss. Universität Zürich, 1955. 37 pp.

The author defines the numbers D_ν by the generating function $3/(e^x+e^{-x}+1)=\sum_{m=0}^\infty D_m x^m/(m!)$, and proves: The class number of the field of 6lth roots of unity is prime to l if and only if the numerators of the first $(l-1)/2$ Bernoulli numbers of even index and the numerators of $D_0, D_2, \cdots, D_{l-3}$ are all prime to l. Using this theorem and the Takagi form of the reciprocity law in the field of 6lth roots of unity, he derives conditions for solvability of the equation mentioned in the title which are analogous to those of Gut [Comment. Math. Helv. **24** (1950), 73–99; **25** (1951), 43–63; MR **12**, 243, 807].

G. Whaples (Bloomington, Ind.)

4538:

Fröhlich, A. On a method for the determination of class number factors in number fields. Mathematika **4** (1957), 113–121.

The author uses the Hilbert theory of ramification groups and local and global class field theory. He asserts that these results contain all known non-analytic results of the type "the class number is not divisible by the prime l" except for a theorem of his [J. London Math. Soc. **29** (1954), 211–217; MR **16**, 573].

Main results: Let K be a finite algebraic number field, h_K the class number (in the narrow sense) and $h_{K,l}$ the power of the prime l dividing h_K. Theorem 1: Let l, p be primes and Ω, K be finite number fields such that K/Ω is normal of degree a power of l, $h_{\Omega,l}=1$, p has precisely one divisor in Ω, and the absolute norm of discriminant of K/Ω is a power of p. Then $h_{K,l}=1$ and p has precisely one prime divisor in K. Theorem 3: If P=rational field, if l, p are odd primes with $p\equiv1 \bmod l$, and if η is a primitive lth root of unity, then the class number of $\mathrm{P}(\eta, p^{1/l})$ is divisible by l. Theorem 5: There exists a cyclotomic extension of the rational field with class number divisible by any given odd integer.

G. Whaples (Bloomington, Ind.)

4539:

Kubota, Tomio. Galois group of the maximal abelian extension over an algebraic number field. Nagoya Math. J. **12** (1957), 177–189.

The character group of the Galois group of the maximal abelian extension of an algebraic number field Ω is described by its Ulm invariants [Kaplansky, Infinite abelian groups, Univ. Michigan Press, Ann Arbor, 1954; MR **16**, 444]. Let l be a prime number; X_l the group of characters whose period is a power of l; L the group of characters of period l, and $L_\nu=L\cap X_l^{l^\nu}$; $X_{l,\infty}'$ the group of all divisible characters in X_l; and $X_{l,\infty}$ the maximal direct product of groups isomorphic to the group of all roots of unity whose period is a power of l which is contained in $X_{l,\infty}'$. The structure of X_l is determined by: (1) dim X_l=number of direct factors of $X_{l,\infty}$; (2) μ_ν= number of direct factors of $L_{\nu-1}/L_\nu$; and (3) $\mu_{\infty,\nu}$=number of cyclic factors of order l^ν in the decomposition of $X_{l,\infty}'/X_{l,\infty}$.

Let ν_l be the largest integer such that Ω contains primitive l^{ν_l}th roots of unity. Let $\mathbf{e}_{l,\nu}$ be the group of all units of Ω which are l^νth powers in the $\mathfrak{l}$-adic completion of Ω for every $\mathfrak{l}|l$. The author proves: There is a μ_l with $l^{\mu_l}=(\mathbf{e}_{l,\nu}:\mathbf{e}_{l,\nu+1})$ for every sufficiently large ν. Then $\mu_\nu=0$ for $\nu<\nu_l$, $\mu_\nu=\infty$ for $\nu\geq\nu_l$, and dim $X_l=N-\mu_l$, where N is the absolute degree of Ω. If l^{c_ν} is the number of elements of $X_{l,\infty}'$ whose orders divide l^ν, then $\mu_{\infty,\nu}=2c_\nu-c_{\nu-1}-c_{\nu+1}$. *G. Whaples* (Bloomington, Ind.)

4540:

Kubota, Tomio. Unit groups of cyclic extensions. Nagoya Math. J. **12** (1957), 221–229.

Let Ω be an algebraic number field of finite degree and l an odd prime not dividing the absolute discriminant of Ω. Then for any group H with $\mathbf{e}^{l^\nu}\subset H\subset\mathbf{e}$, where $\mathbf{e}$ is the unit group of Ω, there is an infinite set of cyclic extensions K, of degree l^ν over Ω, such that $N_K/\Omega\mathbf{e}_K=H$, where $\mathbf{e}_K$ is the unit group of K, and the first cohomology group of $\mathbf{e}_K$ is isomorphic to the direct product of the zero-th cohomology group of $\mathbf{e}_K$ and a cyclic group of order l^ν. This generalizes previous results of the author for the case $\nu=1$ [same J. **9** (1955), 115–118; MR **17**, 714].

G. Whaples (Bloomington, Ind.)

4541:

Dwork, Bernard. Norm residue symbol in local number fields. Abh. Math. Sem. Univ. Hamburg **22** (1958), 180–190.

Let p be a prime number, Q the field of p-adic rationals, and Z a primitive p^n-th root of unity. If u is a unit in Q and $\sigma=(u, Q(Z)/Q)$ is the norm residue symbol then it is known from the global theory that $Z^\sigma=Z^{1/u}$, where $1/u$ is to be interpreted mod p^n. The search for a purely local proof of this explicit cyclotomic reciprocity law has been a problem for many years; it is important because a solution of it would make possible a new proof of the complete global reciprocity law of Artin. The author solves this problem in the following two steps: (a) Find an element of a subfield K of $Q(Z)\cdot T$ (where T is the completion of the maximal unramified extension of Q) whose norm

residue symbol in $Q(Z)\cdot T/K$ is a generator of the Galois group; (b) compute to a good p-adic approximation the norm from K to Q of this element. He uses a new formal power series, related to the Artin-Hasse-Šafarevič E-function. Namely, he uses theorem 1: Let K be an abelian, purely ramified extension of k; Δ, the Frobenius automorphism of KT over K; σ, an element of $G(K/k)$; Π, a prime element of K; then $\sigma^{-1}=(N_{KT/kT}\mathrm{B}, K/k)$, where B is any element in KT such that $\mathrm{B}^{\Delta-1}=\Pi^{\sigma-1}$. Theorem 2: Let K/k be abelian, purely ramified; degree k/Q finite; α an integral element of T such that $(\Delta-1)\alpha=1$, Π a prime element of K, σ an element of $G(k/k)$, $1-t=\Pi^{\sigma-1}$, δ the Frobenius automorphism over Q, and v the largest integer $\leqq 1+(\mathrm{ord}_{\mathfrak{p}}\, p)/(p-1)$; then $N_{K/k}(1+tX)\in 1+X\mathfrak{p}^{v}[X]$ (in the ring of formal power series in X) implies

$$\sigma^{-1}=\Big(\prod_{i=1}^{\infty}(N_{K/k}(1-t\mathfrak{p}^{i}))^{\delta^{i-1}(\delta-1)\alpha/\mathfrak{p}^{i}}, K/k\Big).$$

The author also gives by similar methods a construction for an element $\alpha\in k$ such that $(\alpha, K/k)$ is a specified generator of the Galois group, where K/k is a pure ramified extension of degree p and k is a local field. This is more general than a similar result of MacKenzie and Whaples [Amer. J. Math. **78** (1956), 473–485; MR **19**, 834], because his method uses an arbitrary Eisenstein equation while theirs demanded an Artin-Schreier equation.

G. Whaples (Bloomington, Ind.)

4542:

Nering, Evar D. Reduction of an algebraic function field modulo a prime in the constant field. Ann. of Math. (2) **67** (1958), 590–606.

Let K be an algebraic function field with field of constants k, of finite degree over $F=k(x)$ where $x\in K$ and is transcendental over k. Let h be a discrete non-archimedean valuation of k and extend h to a valuation of $k(x)$ by the Gaussian definition — i.e., value of an element of $k[x]$=maximum value of a coefficient. Suppose that, in the language of divisors, $h=\prod_{i=1}^{s} H_i^{e(H_i)}$ on K. Let K_i, $\bar{k}$ denote the residue class field of K mod H_i and of k mod h. Then K_i is finite algebraic over $\bar{k}(\bar{x})$. Let g, g_i be the genus of K and of K_i. The author proves that if $K\otimes_F F_h$ contains no radical, where F_h is the completion of F under tne valuation h, then

$$g-1=\sum_{i=1}^{s} r_i e(H_i)(g_i-1)+\rho.$$

Here, r_i is the degree of field of constants of K_i over $\bar{k}$ and 2ρ=degree $N_{\bar{K}/F}\mathfrak{f}$: the valuations of $\bar{k}[x]$ which are trivial on $\bar{k}$ can be extended to valuation-like functions ($\bar{K}$-valuations) of the ring $\bar{K}=K^*/hK^*$, where K^* is the set of elements of K which are integral at all H_i, and the ideal $\mathfrak{f}$ (called the conductor of K with respect to $\bar{K}$) is defined by use of the rings of integral elements of $\bar{K}$ with respect to these $\bar{K}$-valuations. The methods and notations are similar to those of Artin [Algebraic numbers and algebraic functions. I, Inst. Math. Mech. New York Univ., New York, 1951; MR **13**, 628].

G. Whaples (Bloomington, Ind.)

4543:

Newman, Morris. Further identities and congruences for the coefficients of modular forms. Canad. J. Math. **10** (1958), 577–586.

The author develops further identities and congruences for $p_r(n)$ defined by $\sum p_r(n)x^n=\prod(1-x^n)$, with $p_r(n)=0$ when not covered by the formula. As in his earlier work [J. London Math. Soc. **31** (1956), 350–359; MR **18**, 194], the author deduces his identities by using rational functions on the Riemann surface for $\Gamma_0(p)$ (where $ad-bc=1$ and $p|c$). The main theorem is

$$p_r(np^2+rv)-\gamma_n p_r(n)+p^{r-2}p_r\Big(\frac{n-rv}{p^2}\Big)=0$$

where p is prime (>2), $\gamma_n=c-((rv-n)/p)p^{(r-3)/2}a$, $c=p_r(rv)+(rv/p)p^{(r-3)/2}a$, $a=\alpha_p\exp\{-i\pi r(p-1)/4\}$, and $\alpha_p=1$ or i depending on whether $p\equiv+1$ or -1 (mod 4). Typical results are $p_{-1}(84n^2-(n^2-1)/24)\equiv 0$ mod 13 for $p_{-1}(n)$ the partition function if $(n, 6)=1$. Also, $p_{15}(53n^2+5(n^2-1)/8)=0$, n odd; and $p_{-1}(13n^2-7)\equiv 6p_{-1}(n)$ mod 13. The paper ends with a table of $p_r(rv)$ ($v=(p^2-1)/24$), for $5\leqq r\leqq 23$, and a table of values of c, found by using the IBM 704 to 15 decimal digit accuracy.

H. Cohn (Tucson, Ariz.)

4544:

Negoescu, N. L'ordination de quelques fractions continues, doubles. An. Şti. Univ. "Al. I. Cuza" Iaşi. Secţ. I (N.S.) **3** (1957), 11–17. (Romanian. Russian and French summaries)

It is known that every irrational number θ can be developed uniquely in a simple infinite continued fraction of the form $\theta=[a_0, a_1, a_2, \cdots]$, where a_0 is an integer and the $a_i\geqq 1$, $i=1, 2, \cdots$, are also integers. One designates by $[a_1, \cdots, a_r, [a_{r+1}], a_{r+2}, \cdots]$ the sum $[a_{r+1}, a_{r+2}, \cdots]+[0, a_r, a_{r-1}, \cdots, a_1]$. This expression is named, after R. Robinson [Bull. Amer. Math. Soc. **53** (1947), 351–361; **54** (1948), 693–705; MR **8**, 566; **10**, 235], a double continued fraction. Using a generalization of the method of Robinson, the author proves, in addition to a number of relations concerning these continued fractions, the following theorem. Let $a_0, a_1, \cdots, a_{r-1}, s$ be integers $\geqq n$; $\alpha, \beta, \gamma, \delta$ real numbers $>n$; B_0 the block $a_{r-1}, \cdots, a_1, [a_0], a_1, \cdots, a_{r-1}$; and r an even number. Then $[\gamma, n, s, B_0, s-1, n, \delta]<[\alpha, n, s, B_0, s, n, \beta]$, $[\gamma, n, s, B_0, s+1, n, \delta]>[\alpha, n, s, B_0, s, n, \beta]$, where $\alpha, \beta<(n^2+1)^2s+n(n^2+n+1)$.

E. Frank (Chicago, Ill.)

COMMUTATIVE RINGS AND ALGEBRAS

See also 4526.

4545:

Veldkamp, G. R. Rings and fields. Nieuw Tijdschr. Wisk. **44** (1956/57), 35–66, 207–235. (Dutch)

Elementary exposition.

4546:

Zierler, Neal. A decomposition theorem for the integers modulo q. Amer. Math. Monthly **65** (1958), 31–32.

Theorem: Let q be a positive integer and let S be the ring of least positive residues modulo q. Let G be the multiplicative group of regular elements of S. Then the collection $\{dG\}_{d|q}$ is a decomposition of S.

A. J. Kempner (Boulder, Colo.)

4547:

Lindenberg, Wolfgang. Die Homogenisierung von algebraischen Funktionenkörpern in n Veränderlichen. Bonn. Math. Schr. no. **5** (1957), 40 pp.

Let k be an arbitrary field and set $K=k[x_1, \cdots, x_n]$ and $\mathfrak{K}=k(x_1, \cdots, x_n)$, where the x_i are indeterminates over k. Let $\mathfrak{L}=\mathfrak{K}(\theta)$ be a separable algebraic extension of $\mathfrak{K}$. Let $\xi_0, \cdots, \xi_n$ be indeterminates over k and set $P=k[\xi_0, \cdots, \xi_n]$, $\Sigma=k(\xi_0, \cdots, \xi_n)$, $x_i=\xi_i\xi_0^{-1}$ and $\Lambda=$

$\Sigma(\theta)$. Let $\bar{P}$ and $\tilde{K}$ be the rings of the elements of Λ and $\mathfrak{L}$, respectively, that are integrally dependent over P and K, respectively. If α is a root of the irreducible equation $\Sigma\,\varphi_i(\xi)z^{M-i}=0$, where φ_i is a quotient of degree ig of homogeneous polynomials, it is said that α is a formal-homogeneous element of Λ of degree g. A homogeneous ideal of $\bar{P}$, or of P, is any ideal which possesses a basis formed by homogeneous elements of $\bar{P}$, or of P, respectively. The main results of this paper are the following: (a) A prime ideal $\bar{\mathfrak{P}}\subset\bar{P}$ is homogeneous if and only if the ideal $\bar{\mathfrak{P}}\cap P$ is a homogeneous one. (b) Every homogeneous ideal of P is the intersection of homogeneous primary ideals. All the minimal prime ideals of a homogeneous ideal are homogeneous ideals.

Proposition (b) is a particular case of theorem 2, page 492 of Zariski [Trans. Amer. Math. Soc. **53** (1943), 490–542; MR **5**, 11], which is not mentioned by the author. The following proposition (2.15 II) of the author appears to be incorrect: There is a one-to-one correspondence between the set of all homogeneous prime ideals, excluding the irrelevant one, of $\bar{P}$ and the set of all prime ideals of the rings $\tilde{K}=\tilde{K}_0, \tilde{K}_1, \cdots, \tilde{K}_n$. For to any prime homogeneous ideal of $\bar{P}$ which is not divisible by any one of the elements $\xi_0, \cdots, \xi_n$ there corresponds one ideal in every one of the rings $\tilde{K}_0, \cdots, \tilde{K}_n$. *P. Abellanas* (Madrid)

4548:

Thierrin, Gabriel. Sur les idéaux fermatiens d'un anneau commutatif. Comment. Math. Helv. 32 (1958), 241–247.

An ideal M in a commutative ring A is $(n\text{-}r)$ fermatian, where n, r are integers such that $n>1$, $r\geqq 1$, if any relation $\sum_{i=1}^{r} a_i^n \in M$ implies $\sum_{i=1}^{r} a_i \in M$. If (0) is $(n\text{-}r)$ fermatian, A is called an $(n\text{-}r)$ fermatian ring. If M is $(n\text{-}r)$ fermatian for all r, M is termed n-fermatian. Fermat's last theorem may be expressed: The ring of integers is not $(n\text{-}r)$ fermatian for any odd n.

The author employs the $(n\text{-}r)$ fermatian radical of an arbitrary ideal M, defined as the intersection of all (n, r) ideals containing M (and shown to be the intersection of all $(n\text{-}r)$ fermatian minimal prime ideals dividing M), to show that every non-trivial $(n\text{-}r)$ fermatian ring is the subdirect sum of $(n\text{-}r)$ fermatian integral domains. Fermat's last theorem for odd n is equivalent to the statement that an infinite number of primes can not be expressed in the form $(a^n+b^n+c^n)/d$, with $a+b+c=1$.

Results analogous to those for $(n\text{-}r)$ fermatian ideals are obtained for n-fermatian ideals. The characteristic of an n-fermatian ring divides q^n-q for all integers q.

{The letter A in theorem 7 should be C.}

J. D. Swift (Los Angeles, Calif.)

4549:

Nöbauer, Wilfried. Über die Operation des Einsetzens in Polynomringen. Math. Ann. 134 (1958), 248–259.

If $\mathfrak{r}$ is a commutative ring with unit, an ideal A of the polynomial ring $\mathfrak{r}[x]$ is defined to be a complete ideal (Vollideale) if and only if $f_1(x)\equiv g_1(x) \bmod A$ and $f_2(x)\equiv g_2(x) \bmod A$ implies $f_1(f_2(x))\equiv g_1(g_2(x)) \bmod A$. Complete ideals are characterized by the fact that an ideal A of $\mathfrak{r}[x]$ is a complete ideal if and only if $u(x)\in A$ and $g(x)\in \mathfrak{r}[x]$ implies $u(g(x))\in A$. If $\mathfrak{a}$ is an ideal of $\mathfrak{r}$, then the ideal $(\mathfrak{a})$ generated in $\mathfrak{r}[x]$ by $\mathfrak{a}$ is a complete ideal, and the set of all polynomials $w(x)$ of $\mathfrak{r}[x]$ for which $r\in\mathfrak{r}$ implies $w(r)\equiv 0 \bmod \mathfrak{a}$ is a complete ideal $\{\mathfrak{a}\}$. It is also shown that for every complete ideal A of $\mathfrak{r}[x]$ there is one and only one ideal $\mathfrak{a}$ of $\mathfrak{r}$, the "including" ideal of A, such that $(\mathfrak{a})\subseteq A\subseteq\{\mathfrak{a}\}$. It is shown that the assignment to each complete ideal of its including ideal is an algebra-homomorphism of the algebra of complete ideals of $\mathfrak{r}[x]$ onto the algebra of ideals of $\mathfrak{r}$.

The semi-group $\mathfrak{H}(A)$ of a complete ideal A is defined in the ring of residue classes $\mathfrak{r}[x]/A$ by the condition that if the residue classes of $u(x)$ and $v(x)$ are $\overline{u(x)}$ and $\overline{v(x)}$ then $\overline{u(x)}\circ\overline{v(x)}=\overline{u(v(x))}$. The group $\mathfrak{R}(A)$ is defined to be the set of invertible elements of $\mathfrak{H}(A)$. Several theorems are given which relate properties of complete ideals to properties of their associated groups and semi-groups. There are also given examples in which the ring $\mathfrak{r}$ is taken to be the ring of rational integers.

H. Levi (New York, N.Y.)

ALGEBRAIC GEOMETRY

See also 4542, 4547, 4829.

4550:

Turri, Tullio. Ipersuperficie invarianti nella trasformazione ottenuta in S_r mediante r polarità. Rend. Sem. Fac. Sci. Univ. Cagliari 27 (1957), 192–195.

Dans un espace S^r, on considère r hyperquadriques Q_i linéairement indépendantes; les hyperplans polaires d'un point P par rapport aux Q_i se coupent en un point P' homologue de P dans une involution T de S^r. Par l'étude de la correspondance entre une droite et la courbe d'ordre r qui lui correspond par T, on montre que les hypersurfaces d'ordre $r+1$ qui passent aux 2^r points communs aux Q_i supposés distincts et par la courbe d'ordre $r(r^{r-2}-1)$ lieu des sommets des cônes du système de quadriques $l_1Q_1+l_2Q_2+\cdots+l_rQ_r$ sont invariantes par T. Les hypersurfaces du système complet ainsi invariant découpent sur la courbe variable intersection de $r-1$ d'entre elles une série spéciale. *B. d'Orgeval* (Dijon)

4551:

Turri, Tullio. Le superficie generali dell' S_3 con una stessa curva di diramazione. Rend. Sem. Fac. Sci. Univ. Cagliari 27 (1957), 196-203.

Les surfaces de S^3 sans singularités représentées par un même plan multiple (même ordre et même courbe de diramation) sont équivalentes par une homologie. Le résultat s'étend au cas d'un nombre fini de points singuliers isolés; dans le cas de courbe double, certaines conditions sont à remplir que l'auteur pense vérifiées dans le cas général. *B. d'Orgeval* (Dijon)

4552:

Spampinato, Nicolò. Sulle V_8 di S_{11} determinate da una falda di Halphen di una superficie completa. Ricerche Mat. 6 (1957), 67–95.

This is the third of a series of papers on Halphen sheets of algebraic surfaces in S_3, i.e., the sheets of a surface at a general point of a multiple curve, each of which contributes one superlinear branch to the general plane section. In the two previous papers [same Ricerche **4** (1955), 191–206; **5** (1956), 226–238; MR **17**, 1240; **18**, 822] the author considered the extension of the geometry by the extension of the ground field of complex numbers. The extension in question is what the author calls tridual numbers, which appear, perhaps, to be numbers of the form $x+ye_1+ze_2$, where $e_1^2=e_2^2=e_1e_2=0$ and x, y, z are

complex numbers. (Unfortunately, the only reference given for this system is to the author's "Lezioni di Geometria Superiore", vol. V [Pironti, Napoli, 1947], to which the reviewer has not been able to obtain access.) The homogeneous coordinates $(x_1, \cdots, x_4)$ of a generic point of the sheet being expressed as analytic functions homogeneous in three parameters (ρ, σ, η), these equations are then split up so as to express the complex components of $(x_1, \cdots, x_4)$, twelve in number, in terms of those of (ρ, σ, η), which are nine in number, and these equations thus define a V_8 in S_{11}, which is then studied in detail. The motivation of this extension of the ground field is somewhat obscure, as the algebra in question is not a field (but a ring with zero divisors) so that the extended geometry is not projective. *P. Du Val* (London)

4553:

Spampinato, Nicolò. Sulla superficie di ordine $(n+v)^2$ che oscula nel punto origine una falda di Halphen di ordine n e classe v. Ricerche Mat. **6** (1957), 195–204.

A sheet of a surface in S_3, along a multiple curve, which contributes a branch of order n and class v to the general plane section, has parametric equations at a general point of the multiple curve, of the form

$$x_1=\rho,\ x_2=a(\rho)+\sigma^n,\ x_3=b(\rho)+c(\rho)\sigma^n+d(\rho)\sigma^{n+v}+\cdots,$$

where $a(\rho)$, $b(\rho)$, $c(\rho)$, $d(\rho)$ are power series in ρ; in these equations the terms involving ρ only are the parametric equations of the multiple curve, and σ is a parameter on the section $x_1=\rho$. Choosing as x_1 axis the tangent to the multiple curve and as plane $x_3=0$ the tangent plane to the sheet, $a(\rho)$, $b(\rho)$ have no constant or linear term and $c(\rho)$ no constant term.

Continuing three previous papers on this subject [see preceding review], the author considers the approximation to this sheet obtained by deleting from these equations all terms of degree $>n+v$ in (ρ, σ), so that $a(\rho)$, $b(\rho)$, $c(\rho)$ are replaced by polynomials $A(\rho)$, $B(\rho)$, $C(\rho)$ (of degrees $n+v$, $n+v$, v respectively) and $d(\rho)$ by its constant term d_0. The result is a rational surface of order $(n+v)^2$, with equations

$$[x_3-B(x_1)-C(x_1)x_2+C(x_1)A(x_1)]^n=d_0{}^n[x_2-A(x_1)]^{n+v}$$

mapped on the (ρ, σ) plane by a linear system of $(n+v)$-ic curves without base points. This has a multiple curve of the same character as that on the given surface, having $(n+v)$ point contact with the latter. It also has the line at infinity in the planes $x_1=\rho$ as $[(n+v)^2-n-v]$-ple curve, the residual sections by these planes being of order $n+v$ and self-dual, having, as well as the branch of order n and class v at the point $(\rho, A(\rho), B(\rho))$, one of order v and class n at infinity on the x_3-axis, touching the line at infinity. *P. Du Val* (London)

4554:

Horadam, A. F. Projection of an invariant locus in [8] from a solid lying on it. Quart. J. Math. Oxford Ser. (2) **9** (1958), 81–86.

In a previous paper [same J. **8** (1957), 241–259; MR **20** #2337] the author considered a locus L of order 45 and dimension 4, lying in S_8. In the present paper the author shows that the projection of L from one of the 81 solids lying on it onto an S_4 is a Burkhardt primal (quartic primal with 45 nodes). Each point of the Burkhardt primal is the projection of a sextic curve of genus 2 lying on L.

J. A. Todd (Cambridge, England)

4555:

Lightstone, A. H.; and Robinson, A. On the representation of Herbrand functions in algebraically closed fields. J. Symb. Logic **22** (1957), 187–204.

Wenn eine Aussage $\mathfrak{X}$ der Form — es werde $\mathfrak{x}$ statt $x_1, \cdots, x_n$ und $\mathfrak{y}$ statt $y_1, \cdots, y_n$ geschrieben — $\forall\mathfrak{x}\exists\mathfrak{y}\, Z(\mathfrak{x}, \mathfrak{y})$ in einer Struktur M^* gilt, so gibt es ein System ϕ von Funktionen $\varphi_1, \cdots, \varphi_m$, den Herbrandfunktionen, so daß $\forall\mathfrak{x}Z(\mathfrak{x}, \phi(\mathfrak{x}))$ in M^* gilt. Für den Fall der algebraisch abgeschlossenen Hülle M^* eines Körpers M zeigen Verff., daß sich die Funktionen φ als stückweise algebraische Funktionen in M^* wählen lassen. Genauer: der Raum $S_n=(M^*)^n$ der $\mathfrak{x}$ läßt sich mit Hilfe algebraischer Mannigfaltigkeiten $V_0\supset V_1\supset V_2\supset\cdots\supset V_r$ so zerlegen: $S_n=(V_0-V_1)\cup(V_1-V_2)\cup\cdots\cup(V_{r-1}-V_r)$ daß für $i=1, \cdots, r$ in $M[\mathfrak{x}, \mathfrak{y}]$ Polynome $p_{i1}(\mathfrak{x}, y_1), \cdots, p_{im}(\mathfrak{x}, y_1, \cdots, y_m)$ mit $P_{ij}(\mathfrak{x}, y_1, \cdots, y_{j-1})$ als höchstem Koeffizienten von p_{ij} in y_j $(j=1, \cdots, m)$ existieren, die für alle $\mathfrak{x}\in V_{i-1}-V_i$

$$\forall\mathfrak{y}[p_{i1}(\mathfrak{x}, y_1)=0\wedge\cdots\wedge p_{im}(\mathfrak{x}, y_1, \cdots, y_m)=0$$
$$\rightarrow P_{i1}(\mathfrak{x})\neq0\wedge\cdots\wedge P_{im}(\mathfrak{x}, y_1, \cdots, y_{m-1})\neq0\wedge Z(\mathfrak{x}, \mathfrak{y})]$$

erfüllen.

Zum Beweis wird die metamathematische Modell-Vollständigkeit der algebraisch abgeschlossenen Körper benützt [A. Robinson, Complete theories, North Holland Publ. Co, Amsterdam, 1956; MR **17**, 817]. Gilt $\mathfrak{X}$ in M^* so auch in der algebraisch abgeschlossenen Hülle $M(t_1, \cdots, t_n)^*$ der Erweiterung von M durch n Unbestimmte. $M(t_1, \cdots, t_n)^*$ enthält daher Elemente $a_1, \cdots, a_m$ mit $Z(t_1, \cdots, t_n, a_1, \cdots, a_m)$. Die Mannigfaltigkeit der Punkte $(\mathfrak{x}, \mathfrak{y})$ mit $Z(\mathfrak{x}, \mathfrak{y})$ ist also durch ein Polynomideal zu charakterisieren, also durch endlich viele Polynome. Der weitere Beweis des Satzes ist einigermaßen kompliziert, insbesondere in der allgemeinen Form für Aussagen $\forall\mathfrak{x}_1\exists\mathfrak{y}_1\forall\mathfrak{x}_2\exists\mathfrak{y}_2\cdots Z(\mathfrak{x}_1, \mathfrak{x}_2, \cdots; \mathfrak{y}_1, \mathfrak{y}_2, \cdots)$.

Alseine Anwendung des Satzes ergibt sich, daß die Funktionen $\varphi_1, \cdots, \varphi_m$ stückweise rational sind, wenn sie eindeutig bestimmt sind. *P. Lorenzen* (Kiel)

4556:

Muracchini, Luigi. Le varietà più volte striate, ed alcune caratterizzazioni delle varietà di Segre. Rend. Mat. e Appl. (5) **17** (1958), 15–34.

Une variété V^k différentiable de classe C^1 est dite striée par un système $S(p)$ si elle contient les espaces projectifs E^p à p dimensions d'un système continu S de dimension $k-p$, en sorte que par tout point générique passe un et un seul E^p du système, et que V ne contienne aucun système de E^p de dimension supérieure à $k-p$. Si la V^k contient s systèmes $S(p_i)$ tels que $p_1+p_2+\cdots+p_s=k$, que les espaces E^{p_i} des systèmes S issus d'un point générique P de V sont indépendants et qu'en P ne passent pas d'autres espaces linéaires situés sur V, alors V^k est dite s-fois striée. Parmi les systèmes $S(p_i)$ deux seront dits permutables si les variétés $P(p_1, p_2)$ et $P(p_2, p_1)$ formées les premières des E^{p_2} issus des points du E^{p_1} issu de P, les secondes des E^{p_1} issus des points du E^{p_2} issu de P, coïncident quel que soit P. La variété $P(p_i, p_k)$ est alors doublement striée. Si une variété est doublement striée elle est algébrique ainsi que ses stries. Si une V^{p+1} est striée par un système ∞^1 de E^p et un système ∞^p de droites, c'est une variété de Segre $W(p, 1)$ ou une de ses projections. Une V^{p+q} deux fois striée par $S(p)$ et $S(q)$ appartient à un espace à $(p+1)(q+1)-1$ dimensions au plus et si ce maximum est atteint c'est une variété de Segre $W(p, q)$. Si une variété V^k $(k=p_1+p_2+1)$ est trois fois striée par les systèmes $S(p_1)$, $S(p_2)$, $S(1)$, elle appar-

tient à un espace de dimension $2(p_1+1)(p_2+1)-1$ et si les systèmes $S(p_1)$ et $S(p_2)$ sont permutables c'est une $W(p_1, p_2, 1)$ de Segre. Si une variété est trois fois striée par des droites et si deux systèmes ne sont pas permutables elle appartient à un système de dimension au plus 6; si elle appartient à un espace à 7 dimensions c'est une variété de Segre. Si une variété est trois fois striée par $S(p_1)$, $S(p_2)$, $S(p_3)$, dont les deux premiers permutables, et appartient à un espace de dimension $(p_1+1)(p_2+1)(p_3+1)-1$, c'est une $W(p_1, p_2, p_3)$ de Segre. Une V^6 trois fois striée par des systèmes de plans et de classe C^∞ appartient à un espace de dimension au plus égale à 26; si cet espace a pour dimension 26, 25, 24 c'est la $W(2, 2, 2)$ de Segre ou l'une de ses projections. Une V^k $(k=p+q+1)$ contenant $S(p)$ et $S(q)$ de dimensions $k-p$ et $k-q$ telle que par un point générique passent un seul E^p et un seul E^q, les deux systèmes n'étant pas permutables, appartient si $p\geqq q$ à un espace de dimension inférieure à $2pq+4q+p+3$.

{Ces résultats obtenus par des méthodes diverses et ingénieuses doivent conduire à de nombreuses généralisations.} *B. d'Orgeval* (Dijon)

4557:

Nakai, Yoshikazu. The existence of irrational pencils on algebraic varieties. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **29** (1955), 151–158.

An algebraic proof of a theorem of de Franchis is given here under the assumption that the characteristic p is 0. The theorem of de Franchis says that a complete nonsingular algebraic variety has an irrational pencil of divisors if and only if there is a linear differential form of the first kind of the type $f\cdot dg$, where f and g are rational functions on V.

The following two theorems are proved as applications of the theorem. (i) Let w be a linear differential form of the first kind on V. If $l((w))>1$, V has an irrational pencil of divisors. (ii) Abelian varieties cannot have irrational pencils of genus >1. {Reviewer's note: (ii) is still true even when $p>0$, because a rational map from an Abelian variety into an Abelian variety is essentially a homomorphism.} *T. Matsusaka* (Evanston, Ill.)

4558:

Morin, Ugo. Problemi di razionalità ed analisi indeterminata. Rend. Sem. Mat. Fis. Milano **27** (1957), 160–166.

The author shows how the methods of classical diophantine analysis may be applied to study the existence of a surface which unisects the rational curves of a system of curves on a variety $V_3{}^n$ in S_4.

G. B. Huff (Athens, Ga.)

4559:

✱Serre, Jean-Pierre. Sur la topologie des variétés algébriques en caractéristique p. Symposium internacional de topología algebraica [International symposium on algebraic topology], pp. 24–53. Universidad Nacional Autónoma de México and UNESCO, Mexico City, 1958. xiv+334 pp.

For any commutative ring A of characteristic $p>0$ form the ring $W_n(A)$ of Witt vectors of length n; this consists of all n-tuples $(a_0, \cdots, a_{n-1})$ of elements of A together with two (polynomial) laws of composition that make $W_n(A)$ a commutative ring. The Frobenius endomorphism F (raising everything to the pth power) operates on each $W_n(A)$; there is a restriction homomorphism $R:W_{n+1}(A)\to W_n(A)$ (cutting off the last component of the vector), and an (additive) homomorphism V: $W_n(A)\to W_{n+1}(A)$ (adding a zero in front). If X is a variety over the algebraically closed field k of characteristic $p>0$, for each $x\in X$ form $W_n(\mathcal{O}_x)$, $\mathcal{O}_x$ being the local ring at x; all these $W_n(\mathcal{O}_x)$ fit together into the sheaf (of rings) $\mathscr{W}_n$ of Witt vectors of length n on X, and we have obvious extensions of F, R, V to these sheaves. ($\mathscr{W}_1$ is merely $\mathcal{O}$, the sheaf of local rings on X.) The homology groups $H^q(X, \mathscr{W}_n)$ possess familiar properties, for example they vanish if $q>\dim X$ or if X is affine and $q>0$, and are finite modules over $W_n(k)$ if X is projective. For $n\geqq m$, corresponding to the exact sequence

$$0\to\mathscr{W}_m \xrightarrow{V^{n-m}} \mathscr{W}_n \xrightarrow{R^m} \mathscr{W}_{n-m}\to 0,$$

there is an exact cohomology sequence which gives, in particular, Bockstein operations

$$\delta_{n,m}{}^q: H^q(X, \mathscr{W}_{n-m})\to H^{q+1}(X, \mathscr{W}_m).$$

If, for fixed q, all $\delta_{n,m}{}^q$ are zero, X is said to be (homologically) torsion free in dimension q; X is always torsion free in dimensions 0 and $\dim X$, but torsion may exist in other dimensions. For fixed q, the various $H^q(X, \mathscr{W}_n)$ form (via R) a projective system whose limit H^q is a module over $\Lambda=\lim W_n(k)$, a valuation ring of characteristic zero. If X is projective, H^q need not be a Λ-module of finite type, even if $q=1$ (example: curve with a cusp), in which case finite module implies free module; but if X is projective and torsion-free in dimensions $q-1$ and q and if the endomorphism F on $H^q(X, \mathcal{O})$ is surjective, then H^q is free of rank $\dim H^q(X, \mathcal{O})$. For X projective, let σ be the dimension of the part of $H^1(X, \mathcal{O})$ on which F is bijective; for X an irreducible nonsingular curve of genus g this operation of F on $H^1(X, \mathcal{O})$ amounts to the so-called Hasse-Witt matrix of X, and H^1 is a free Λ-module of rank $2g-\sigma$, p^σ turning out (via use of the Cartier operation on differentials) to be the number of divisor classes on X of order p.

The final part of the paper concerns coverings. If the finite group G operates on the variety Y, each orbit being contained in an affine open subset of Y, then there exists a quotient variety $X=Y/G$. If G operates on Y without fixed points, Y is called a G-covering of X; the set of all G-coverings of X is denoted by $\pi^1(X, G)$. For $Y\in\pi^1(X, G)$ and a homomorphism $f:G\to H$, one obtains a variety $Y\times_G H\in\pi^1(X, H)$ (if H is finite); if G is abelian one gets an abelian group structure on $\pi^1(X, G)$. If H is an algebraic group, $Y\times_G H$ has the structure of a principal fiber space over X with group H, minus (possibly) the condition of local triviality, a condition which is filled if H is an algebraic subgroup of GL_n for which there exists a rational cross section $\mathrm{GL}_n/H\to\mathrm{GL}_n$, in particular if H is GL_n, SL_n, Sp_n or connected and solvable. Embedding $\mathbf{Z}/p^n\mathbf{Z}$ in the additive group $W_n(k)$ and using the relation between H-principal fiber spaces over X and elements of $H^1(X, \mathscr{H})$, it is proved that, for X projective, $\pi^1(X, \mathbf{Z}/p\mathbf{Z})$ is finite of order p^σ and, if X is torsion-free in dimension 1, $\pi^1(X, \mathbf{Z}/p^n\mathbf{Z})$ is the direct sum of σ copies of $\mathbf{Z}/p^n\mathbf{Z}$. Finally, there is an example of a nonsingular projective surface with $h^{0,1}\neq h^{1,0}$. *M. Rosenlicht* (Evanston, Ill.)

4560:

Roquette, Peter. Zur Theorie der Konstantenreduktion algebraischer Mannigfaltigkeiten. Invarianz des arithmetischen Geschlechts einer Mannigfaltigkeit und der virtuellen Dimension ihrer Divisoren. J. Reine Angew. Math. **200** (1958), 1–44.

Let the projective variety V be defined over a field with an arbitrary valuation; reduce modulo the valuation those elements of the ideal of V having finite coefficients, to get an ideal over the residue field; assume that the reduced ideal is prime; and so get a projective variety $\bar{V}$ over the residue field. This is the very general context for which arithmetic proofs of the title results are given. (For the second result, as well as for the existence of a natural homomorphism from the divisor class group of V into that of $\bar{V}$, it is also assumed that $\bar{V}$ is absolutely irreducible and absolutely nonsingular.)

M. Rosenlicht (Evanston, Ill.)

4561:

Nishi, Mieo. On the imbedding of a non-singular variety in an irreducible complete intersection. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **29** (1955), 177–187.

If V^r is a nonsingular subvariety of a projective space of dimension $\geqq 2r$, there exists a nonsingular U^{2r} which contains V and is a complete intersection of hypersurfaces. On the other hand, there exists a nonsingular cubic surface in projective 4-space which does not lie on any nonsingular hypersurface.

M. Rosenlicht (Evanston, Ill.)

4562:

Serre, Jean-Pierre. Quelques propriétés des variétés abéliennes en caractéristique p. Amer. J. Math. **80** (1958), 715–739.

If A^g is an abelian variety and $\mathcal{O}_A$ its sheaf of local rings, then $H^*(A)=\sum H^n(A, \mathcal{O}_A)$ is a Hopf algebra; from the structure theorem of Borel and the knowledge of $H^1(A, \mathcal{O}_A)$ it follows that $H^*(A)$ is the exterior algebra of a vector space of dimension g. Most further results concern the case in which the base field is an algebraically closed field k of characteristic $p>0$. It is shown that A has no homological torsion, i.e., its Bockstein operations all vanish [cf. #4559 above]. An example is given of an abelian variety A^2 for which the Witt vector cohomology group $H^2(A, \mathscr{W})$ is not a finite module (over the ring of Witt vectors $\Lambda=W(k)$, cf. loc. cit.). The next part concerns purely inseparable isogenies of height one of a connected algebraic group G, i.e., homomorphisms $G\to G'$ such that $k(G)\supset k(G')\supset (k(G))^p$; the purely inseparable galois theory of Jacobson is used to obtain a one-one correspondence between classes of such isogenies and the p-subalgebras of the Lie algebra of G that are invariant under the adjoint representation. If $f: V\to G$ is a rational map of an irreducible variety into a commutative algebraic group, it is impossible to obtain a factorization $V\to G_1\to G$, where $G_1\to G$ is a nontrivial purely inseparable isogeny, if and only if for each nonzero invariant differential ω of degree 1 on G one has $f^*(\omega)\neq 0$. For commutative algebraic groups A, B, the group of classes of commutative algebraic extension groups Ext (A, B) is discussed, in particular, its functorial properties with respect to an inseparable isogeny $A\to A/\mathfrak{n}$, where A is connected and $\mathfrak{n}$ is a p-subalgebra of its Lie algebra. For A an abelian variety and B connected and linear, Ext (A, B) is isomorphic to a subgroup of $H^1(A, \mathscr{B})$, $\mathscr{B}$ being the sheaf of germs of regular maps from A to B, and the elements of this subgroup are characterized; in particular, for the additive Witt group W_n, Ext $(A, W_n)=H^1(A, \mathscr{W}_n)$, and Ext (A, G_m) is the Picard variety of A. Any isogeny $A\to A'$ is shown to be factorizable into isogenies of four types: separable of degree prime to p, or p, or purely inseparable and corresponding to a Lie algebra element ξ such that $\xi^p=0$, or $\xi^p=\xi$. The homological properties of isogenies of the several types are discussed, which leads to a proof that an abelian variety has no torsion, i.e., for a divisor X on A, "$X\equiv 0$" is equivalent to "X is algebraically equivalent to 0". Finally, for an abelian variety A, let $L_p(A)$ denote the direct sum of the Λ-module $H^1(A, \mathscr{W})$ and of the module obtained from the Tate group of the Picard variety of A (i.e., the group of all sequences $(x_1, x_2, x_3, \cdots)$ of elements of the Picard variety such that $px_1=0$, $px_2=x_1$, $px_3=x_2$, $\cdots$) by extending coefficients to Λ; $L_p(A)$ is a free Λ-module of rank 2 dim A and any endomorphism of A has the same characteristic polynomial as the induced endomorphism of $L_p(A)$, giving a p-adic analogue of Weil's l-adic representations.

M. Rosenlicht (Evanston, Ill.)

4563:

Gallarati, Dionisio. Ancora sulla differenza tra la classe e l'ordine di una superficie algebrica. Ricerche Mat. **6** (1957), 111–124.

The author is concerned with the order n and class ν of an algebraic surface, either without singularity or with only ordinary singularities. It was shown by Marchionna [Boll. Un. Mat. Ital. (3) **10** (1955), 478–480; MR **17**, 1136] that $\nu-n\geqq -1$, with equality only for the plane and the Veronese surface, and by the present author [Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. (8) **21** (1956), 55–56; MR **18**, 671] that $\nu-n=0$ if and only if the surface is ruled but not developable. It is now shown, by manipulation of standard formulae, that except in these cases $\nu-n\geqq 3$, and that the only cases in which $3\leqq \nu-n\leqq 10$ are the del Pezzo surfaces ($\nu=12$, $3\leqq n\leqq 9$) and rational surfaces with hyperplane sections of sums 2 ($\nu=20$, $10\leqq n\leqq 12$); also that every value of $\nu-n\geqq 3$ is realisable for a rational surface. *P. Du Val* (London)

4564:

Thalberg, Olaf M. 'Conic involutions' with a coincident curve of order $4n$. Avh. Norske Vid. Akad. Oslo. I. **1957**, no. 2, 16 pp.

A "conic involution" is defined geometrically and shown to be a Cremona involution I_{10} of order 10 with nine fundamental points. The involution is studied completely and a generalization is given to an involution I_{8n+2} with $4n+5$ fundamental points. *G. B. Huff* (Athens, Ga.)

4565:

Bottema, O. A Cremona transformation of the fifth degree in the plane. Simon Stevin **32** (1958), 61–67. (Dutch)

Four points A_1, A_2, A_3 and O are given in the plane π. A point P in π determines a conic through A_i and tangent at P to OP. When we study the correspondence between P and the point of tangency P' of the second tangent from O to the conic, we find that it is involutory and birational of degree five. There are three singular points, A_i, and three singular conics. The relations between the characteristic numbers of a curve of degree n and its image are established. The correspondence is the product of three quadratic correspondences.

D. J. Struik (Cambridge, Mass.)

LINEAR ALGEBRA

See also 4724.

4566:

Egerváry, Jenő. On a lemma of Stieltjes. Mat. Lapok **7** (1956), 271–276. (Hungarian)

Hungarian translation of Acta Sci. Math. Szeged. **15** (1954), 99–103 [MR **15**, 671].

4567:

***Zurmühl, Rudolf. Matrizen: eine Darstellung für Ingenieure.** 2te Aufl. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1958. xv+467 pp. DM 33.00.

The first, 1950, edition of this book was reviewed in MR **12**, 73; the second edition has been enlarged by forty pages and fifty illustrations. This increase in size represents not only an addition of new material, but also considerable revision and improvement in the presentation of old. The principal change would seem to be in the direction of greater rigour, although the book is still eminently readable by engineers and physicists. The chapter on numerical methods has been brought somewhat up to date; the method of conjugate gradients is included, but an account of other recent developments, such as those due to Lanczos, Givens, Householder and others, would also have been appreciated by some readers.

4568:

Mitchell, B. E. Another note on quasi-idempotent matrices. Amer. Math. Monthly **64** (1957), 716–717.

The author shows that certain properties of quasi-idempotent matrices [G. B. Huff, same Monthly **62** (1955), 334–339; MR **16**, 989] can be studied clearly and simply by using the Jordan canonical form. The idea is illustrated by classifying all quasi-idempotent matrices of order 3.

G. B. Huff (Athens, Ga.)

4569:

Robinson, D. W. An application of the decomposition of a matrix into principal idempotents. Amer. Math. Monthly **65** (1958), 694–695.

The application is "a proof of the following well-known result: if the nth derivative of a function f exists at α, then it can be computed as the limit of

$$h^{-n}\sum_{m=0}^{n}\binom{n}{m}(-1)^m f(\alpha+(n-m)h)$$

as h approaches zero." *C. Davis* (Providence, R.I.)

4570:

Bellman, Richard. Notes on matrix theory. XIV. On the Jacobi relation for the bracket symbol. Amer. Math. Monthly **65** (1958), 605–606.

The familiar fact that the Jacobi identity is satisfied by the commutator $[A, B]=AB-BA$, for square matrices A, B, is obtained as follows. In the identity $e^A(e^Be^C)=(e^Ae^B)e^C$, the exponential is expanded in power series; cyclic permutations are added; the third-degree terms are compared. *C. Davis* (Providence, R.I.)

4571:

Böhm, Corrado. Sulla minimizzazione di una funzione del prodotto di enti non commutativi. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **23** (1957), 386–389.

In connection with multi-stage production processes, e.g., "bottleneck processes", the problem of determining a sequence of matrices to maximize functions of the elements in the product arises in a natural way. The author presents some results pertaining to this and similar problems involving non-commutative quantities.

R. Bellman (Santa Monica, Calif.)

4572:

Lax, P. D. Differential equations, difference equations and matrix theory. Comm. Pure Appl. Math. **11** (1958), 175–194.

The main aim of the paper is the study of the following three theorems concerning sets of $n\times n$ matrices with real elements which span (by real combinations) a space of matrices $\mathfrak{X}$, all of which have real eigen-values. (It is shown that such a set is not necessarily similar to a set of real symmetric matrices.) Let the eigen-values $\lambda_i(X)$ be arranged in increasing order. A partial order is introduced in $\mathfrak{X}$ by saying that $X\leq Y$ whenever $Y-X$ has non-negative eigen-values. 1. $\lambda_n(X)$ is convex and $\lambda_1(X)$ is concave. 2. If $X\in\mathfrak{X}$, $Y\in\mathfrak{X}$ and $X<0$, then the eigen-values of $X+iY$ have negative real parts. 3. $\lambda_i(X)$ is a monotone function of X. (Theorems 1 and 3 follow from the classical min-max principle in the symmetric case.) Particularly interesting is that the proofs of these theorems are based on results from the theory of hyperbolic differential equations: (1) $u_t=\sum A^k u_{x^k}$ where the A^k are real constant $n\times n$ matrices and u is a vector of n unknown functions. The condition that the A^k $(k=1,\cdots,m)$ should span a space $\mathfrak{X}$ is necessary and sufficient for the initial value problem for (1) to be properly posed. An alternative proof of 1, 3 would follow from the symmetric case, granted the truth of the following conjecture of the author and Chandler Davis: if $p(\xi,\eta,\lambda)$ is a form of degree n in ξ,η,λ with the coefficient of λ^n unity which has only real zeros in λ for every fixed real choice of ξ,η, then there are symmetric matrices A, B such that $p(\xi,\eta,\lambda)=\det(\xi A+\eta B-\lambda I)$. Another consequence of the truth of this (algebraic) conjecture which is discussed is the possibility of construction of energy integrals for hyperbolic equations. {The question whether a general form with complex coefficients $p(\xi,\eta,\lambda)$ with 1 as coefficient of λ^n can be represented as $\det(\xi A+\eta B-\lambda I)$ for suitable complex $n\times n$ matrices was studied previously [see T. G. Room, The geometry of determinantal loci, University Press, Cambridge, 1938; for general theorems about $\det(\xi A+\eta B-\lambda I)$ cf. also T. S. Motzkin and O. Taussky, Trans. Amer. Math. Soc. **80** (1955), 387–401; MR **19**, 242].} The matrix results are applied to study the convergence as $\Delta t\to 0$ of the solutions of a difference equation $u(x, t+\Delta t)=\sum C^j u(x+\Delta_j, t)$ (where $\Delta_j=r_j\Delta t$, with the r_j fixed vectors) to the solution of (1). The von Neumann criterion [see, e.g., P. D. Lax and R. D. Richtmyer, Comm. Pure Appl. Math. **9** (1956), 267–293; MR **18**, 48], which is necessary and sufficient for convergence, is that the eigen-values of the amplification matrix $G(\xi)=\sum C^j e^{ir_j\xi}$ (which indicates the behaviour of an initial exponential function) should remain within the unit circle for all real ξ. Since this condition is very difficult to verify in practical cases, a more convenient sufficient condition is established. This is a generalization of one obtained in the symmetric case by Friedrichs [Comm. Pure Appl. Math. **7** (1954), 345–392; MR **16**, 44]. This new condition is that all the C^j have non-negative eigen-values and that they generate a space $\mathfrak{X}$. Since the C^j are usually linear combinations of the A^j and the latter generate a space $\mathfrak{X}$,

the problem being properly posed, part of this condition is automatically satisfied.

O. Taussky-Todd (Pasadena, Calif.)

4573:

Weinberger, H. F. Remarks on the preceding paper of Lax. Comm. Pure Appl. Math. **11** (1958), 195–196.

The matrix theorems of the paper by P. D. Lax [#4572 above] are re-proved here, even in generalized form, by matrix methods only.

O. Taussky-Todd (Pasadena, Calif.)

ASSOCIATIVE RINGS AND ALGEBRAS

4574:

Barsotti, Iacopo. Noncountable normally locally finite division algebras. Proc. Amer. Math. Soc. **8** (1957), 1101–1103.

An algebraic division algebra A over a field F is centrally (= normally) locally finite if every finite subset of elements of A is contained in a subalgebra of finite dimensions and central over F. Such algebras of countable infinite dimensions first seem to have been studied by G. Köthe [Math. Ann. **105** (1931), 15–39], who characterized them as direct products of countably many central division algebras, each of finite dimensions. In Rend. Mat. e Appl. (5) **7** (1948), 1–30 [MR **10**, 179], the author established the existence of centrally locally finite algebras of uncountable dimensions over certain fields, called regular fields, of arbitrary uncountable cardinality. In the present note the same construction as before is combined with a simpler technique to provide such algebras over regular fields of arbitrary (infinite) cardinality. After observing that his construction always leads to algebras of $\aleph_1$ dimensions, the author raises the question of the existence of such algebras of larger dimensions.

C. C. Faith (University Park, Pa.)

4575:

Moriya, Mikao; Nagahara, Takasi; and Tominaga, Hisao. A note on Galois theory of division rings. Math. J. Okayama Univ. **7** (1957), 83–88.

Let K be a division ring with center C, and let M be a maximal (commutative) subfield of K. $G(K/M)$ is the Galois group (=total group) of K/M. For a subset S of K, $V_K(S)$ is the centralizer of S in K. In this paper, under the assumption (a) K/M is locally finite and two rather complicated assumptions (b–c) (which need not be reproduced here), the authors develop a Galois theory in which the correspondence is 1-1 between the intermediate subfields M' of M/C and the intermediate (division) subrings $V_K(M')$ of K/M, $M'=V_K(V_K(M'))$. By remarking that the "regular" (=complete) subgroups of $G(K/M)$ which are closed in Krull's (finite) topology on $G(K/M)$ are themselves Galois groups, the authors are able to express this result in terms of the subrings of K/M and closed regular subgroups of $G(K/M)$. Although the result is well-known then, assumptions (a–c) are trivially fulfilled when K/C has finite dimension. The authors show that they are also satisfied by an important class of infinite dimensional division algebras first constructed by G. Köthe. These are the infinite Kronecker products of central division algebras of relatively prime degrees over the field C of rational numbers.

C. C. Faith (University Park, Pa.)

4576:

✱Bourbaki, N. Éléments de mathématique. 23. Première partie: Les structures fondamentales de l'analyse. Livre II: Algèbre. Chapitre 8: Modules et anneaux semisimples. Actualités Sci. Ind. no. 1261. Hermann, Paris, 1958. 189 pp. (1 insert) 2000 francs.

This book is an exposition of a fundamental theory of semi-simple rings with minimum condition for left ideals, with a glance at the general case. The notions are introduced mainly in terms of modules. For instance: A semisimple module is a direct sum of simple modules and a ring A is semi-simple if A, regarded as a left A-module, is a semi-simple module. (Observe that this condition of semi-simplicity for a ring implies the minimum condition because the existence of the identity is assumed for any ring.) The radical of a ring is defined in the sense of Perlis-Jacobson (so-called "Jacobson radical"). Furthermore, some notions in the general case, such as primitive rings, regular rings, etc., are treated in exercises.

{Of course, there are many results in the theory of algebras which are not stated in this book; see Nakayama-Azumaya, "Algebra II" (in Japanese) [Iwanamishoten, Tokyo, 1954; MR **16**, 895], and Jacobson, "Structure of rings" [Amer. Math. Soc. Colloq. Publ. vol. 37, Providence, R.I., 1956; MR **18**, 373].}

The notion of commutors is introduced in § 1. Artin modules, Noetherian modules, (left) Artin rings and (left) Noetherian rings are defined in § 2. Simple modules and semi-simple modules are defined in § 3. Some results on the commutor of a semi-simple module, which are fundamental for the structure theorem, are proved in § 4. In § 5, simple or semi-simple rings are defined, their structure (as matric rings) is shown and semi-simple sub-algebras of a semi-simple algebra are studied. The Jacobson radical is defined in § 6, and it is proved that the radical of an Artin ring is nilpotent and that an Artin ring is Noetherian. The subject is restricted, from § 7 to the end, to algebras over a (commutative) field. In § 7, variance of radicals and semi-simplicity in tensor products compared with those of factors are studied and the notion of separability of modules or of rings is introduced. Some remarks on composite of extension fields are given in § 8. Semisimple sub-algebras of the ring of endomorphisms of a finite dimensional vector space over a field are observed in § 9. A well-known theory of simple algebras, on inner automorphisms, on Brauer groups (the group of simple algebras with center K modulo matric rings $\mathbf{M}_n(K)$, product being tensor product) and on splitting fields (which are called "corps neutralisants" in this book), is introduced in § 10. Commutativity of finite fields and a characterization of the quaternion field are the topics in § 11. The notions of norm and trace are introduced in § 12. The theory of representation of algebras is introduced a little in § 13. In the appendix, imbedding of an algebra without identity into one with identity is discussed. The historical note at the end of the book contains a history until around 1930.

M. Nagata (Cambridge, Mass.)

4577:

Martindale, Wallace S., III. The structure of a special class of rings. Proc. Amer. Math. Soc. **9** (1958), 714–721.

A ring R is a ξ-ring if for every $x \in R$ there exists $c(x) \in R$ such that $x^2c(x)-x \in Z$, the center of R. The main result is that every ξ-ring is a subdirect sum of subdirectly irreducible ξ-rings R_α, where either R_α is a division algebra or every commutator of R_α lies in the center of R_α. If R is an algebraic algebra over a perfect

field Φ, then each R_α is either a division algebra or commutative; if Φ is not perfect only the following is true: if R_α is not a division algebra and not commutative, then it contains an ideal P_α belonging to its center and R_α/P_α is a field. *S. A. Amitsur* (Notre Dame, Ind.)

4578:

Szép, J. Über eine neue Erweiterung von Ringen. I. Acta Sci. Math. Szeged **19** (1958), 51–62.

Let R be a ring containing two subrings A and B, with $R=A+B$ and $A\cap B=0$. Then the additive group of R is a direct sum of the additive groups of A and B. For $a\in A$, $b\in B$, maps from $A\times B$ to A, $A\times B$ to B, $B\times A$ to A, and $B\times A$ to B are defined by $a\cdot b=a^b+{}^ab$ and $b\cdot a={}^ba+b^a$. A and B together with these maps entirely determine the structure of R; the author writes $R=A\dot{+}B$ for this situation. He then considers two rings A and B together with maps as above. He gives necessary and sufficient conditions on these maps that $A\oplus B$, with $(a,b)\cdot(a_1,b_1)=(a\cdot a_1+a^{b_1}+{}^ba_1,\ b^{a_1}+{}^ab_1+b\cdot b_1)$, be a ring. Thus, in terms of the existence of certain maps, he constructs all rings R with $R=A^1\dot{+}B^1$, $A^1\simeq A$, $B^1\simeq B$.

D. K. Harrison (Haverford, Pa.)

4579:

Szendrei, J. Über eine allgemeine Ringkonstruktion durch schiefes Produkt. Acta Sci. Math. Szeged **19** (1958), 63–76.

The author considers two rings R and P together with eight maps; the notation and domains for these maps will be clear from what follows. He gives necessary and sufficient conditions on the maps that the set of all ordered pairs (a,α), $a\in R$, $\alpha\in \mathrm{P}$, form a ring with $(a,\alpha)+(b,\beta)=(a+b+[\alpha,\beta],\ [a,b]+\alpha+\beta)$ and $(a,\alpha)(b,\beta)=(ab+a^\beta+{}^\alpha b+\{\alpha,\beta\},\ \{a,b\}+\alpha^b+{}^a\beta+\alpha\beta)$. The conditions are simple, but extremely numerous. By restricting certain of the maps to be zero, this main result gives theorems of Everett [Amer. J. Math. **64** (1942), 363–370; MR **4**, 69] and of Szép [see #4578 above]. Other interesting special cases are considered. *D. K. Harrison* (Haverford, Pa.)

4580:

Vidav, Ivan. Le spectre du produit a^*a de deux éléments a et a^* verifiant la relation $aa^*-a^*a=e$. Glasnik Mat.-Fiz. Astr. Društvo Mat. Fiz. Hrvatske Ser. II. **12** (1957), 3–7. (Serbo-Croatian summary)

Let A be a complex algebra with an identity element e and an involution $x\to x^*$. A linear functional f on A is called "positive" if $f(x^*x)\geqq 0$ for all $x\in A$. The author defines the "spectrum" of an element $a\in A$ to consist of all complex numbers λ with the property that, for arbitrary $\varepsilon>0$ and positive integer n, there exists a positive functional f with $f(e)=1$ such that $f((a-\lambda e)^{*n}(a-\lambda e)^n)<\varepsilon$. In particular, if f exists such that $f((a-\lambda e)^*(a-\lambda e))=0$, then λ is in the spectrum and is said to belong to the "discrete spectrum" of a. All other spectral points constitute the "continuous spectrum". A point of the discrete spectrum whose associated positive functional is uniquely determined is called "simple".

Now assume that the algebra A is generated by two elements a and a^* such that $aa^*-a^*a=e$. In this situation, the author obtains the following result: The continuous spectrum of a^*a is empty and the discrete spectrum coincides with the set of non-negative integers each of which is a simple point. Every complex number belongs to the discrete spectrum of a. Analogous results were obtained by C. R. Putnam [J. London Math. Soc. **29** (1954), 350–354; MR **16**, 146] for certain operators on Hilbert space. *C. E. Rickart* (New Haven, Conn.)

4581:

Sulin'skiĭ, A. Some questions in the general theory of radicals. Mat. Sb. N.S. **44(86)** (1958), 273–286. (Russian)

If K is a class of rings with an operator domain, if K_S is the class of simple rings in K, and if P is a property of a ring such that K has a P-radical, Kurosh [same Sb. **33(75)** (1953), 13–26; MR **15**, 194] partitioned K_S into the upper class K_U, consisting of simple P-semisimple rings, and the lower class K_L, consisting of simple P-radical rings; he also reversed the problem to study the radicals, such as the upper U and the lower L, obtained from a given partition of K_S into two subclasses. A partition is called regular provided every U-semisimple ring is a subdirect sum of simple rings belonging to K_U. For the case when K is the class of all associative rings with an operator domain, if a partition of K_S has the property that there are in K_U no simple rings without identity, then the partition is regular, and furthermore every ideal of a U-radical ring is a U-ideal. The converse holds for the case when K is the class of all associative rings without operators. For the case of the class of all non-associative algebras over a fixed field, a necessary condition for the regularity of the partition is that the upper class contain no algebras without identity. For the case when K is the class of all associative algebras of finite rank over a fixed field, the discussion is separated into two subcases according as the algebra Z is a radical ring or a semisimple ring, where Z is the algebra of rank one with all products zero. The class T of all associative algebras (over a fixed field) of finite rank with an identity is shown to be a radical class, and the various radicals contained in the radical T are studied in detail. Another topic is which P-radicals in K have the property that every ideal in a P-radical ring is a P-ideal. Finally the results are applied to the class of all commutative associative algebras of finite rank over a fixed field. *R. A. Good* (College Park, Md.)

NON-ASSOCIATIVE RINGS AND ALGEBRAS

See also 4581.

4582:

Kostrikin, A. I. On local nilpotency of Lie rings that satisfy Engel's condition. Dokl. Akad. Nauk SSSR (N.S.) **118** (1958), 1074–1077. (Russian)

Every Lie ring which satisfies the nth Engel condition and whose characteristic is either 0 or else $p>n+[n/2]$ is locally nilpotent. [See Kostrikin, Izv. Akad. Nauk SSSR Ser. Mat. **21** (1957), 515–540; MR **20** #1701].

R. A. Good (College Park, Md.)

4583:

Cartier, P. Remarques sur le théorème de Birkhoff-Witt. Ann. Scuola Norm. Sup. Pisa (3) **12** (1958), 1–4.

M. Lazard [Publ. Sci. Univ. Alger. Sér. A **1** (1954), 281–294; MR **17**, 645] has reformulated the Birkhoff-Witt theorem in terms of the graded algebra of tensors of a Lie algebra over a commutative ring A. The author shows that the theorem in question is valid if A is a Dedekind ring, and has simplified a counterexample of Lazard showing that the result is not true in general: it is false for A the exterior algebra with three generators over the Galois field J_2. *G. Birkhoff* (Cambridge, Mass.)

4584:

Campbell, H. E. On the Casimir operator. Pacific J. Math. 7 (1957), 1325–1331.

The author extends to certain Lie and alternative (including associative) algebras of arbitrary characteristic results which have been obtained for characteristic 0 by means of a Casimir operator [Amer. J. Math. **64** (1942), 677–694; Proc. Amer. Math. Soc. **73** (1953), 444–451; MR **4**, 71; **15**, 6]. A Lie or alternative algebra A is called nondegenerate by the author in case the bilinear form trace $(R_x R_y)$ is nondegenerate; this implies that A is a direct sum of simple algebras. Let $e_1, \cdots, e_n$ be a basis for A, and let $e_1', \cdots, e_n'$ be the complementary basis such that trace $(R_{e_i} R_{e_j'}) = \delta_{ij}$. Let $x \to S_x$ be a representation of A (where, if A is alternative, the S_x part of a (bi)representation $x \to (S_x, T_x)$ is meant). The author defines a Casimir operator for this representation by $\Gamma_S = \sum_{i=1}^n S_{e_i} S_{e_i'}$. Then $\Gamma_S = S_e$, where $e = \sum e_i e_i'$ is the identity element of A. The first Whitehead lemma for nondegenerate alternative algebras is proved (equivalently, if A is a nondegenerate subalgebra of an alternative algebra B, then any derivation of A into B can be extended to an inner derivation of B); see the related, but independent, paper of Taft [Proc. Amer. Math. Soc. **8** (1957), 950–956; MR **19**, 728]. Also, the Levi theorem holds for a very special case of Lie algebras of prime characteristic: if L is a Lie algebra with "radical" $R \neq L$ such that $LR = 0$ and L/R is nondegenerate, then there is a subalgebra S of L such that $S \cong L/R$ (and L is the direct sum $L = S + R$). *R. D. Schafer* (Princeton, N.J.)

4585:

✱Raffin, Raymond. Remarques sur certaines algèbres de Lie. Symposium internacional de topología algebraica [International symposium on algebraic topology], pp. 83–86. Universidad Nacional Autónoma de México and UNESCO, Mexico City, 1958. xiv+334 pp.

Let A be a non-associative algebra over a field F. Denote by (x, y) the vector space spanned by x and y over F. The following results are announced. If $xy \in (x, y)$ for all x, y in A, then A is power-associative; if, moreover, A is commutative, then A is a Jordan algebra. If $xy \in (x, y)$ for all x, y in A, and if for every $u \neq 0$ in A there is an element in A which is linearly independent of and anticommutative with u, then A is a Lie algebra. If $x(yz) \in (y, z)$, $x^2 = 0$ for all x, y, z in A and if dim $A > 2$, then A is a Lie algebra. *R. Ree* (Vancouver, B.C.)

4586:

Taft, E. J. Invariant Wedderburn factors. Illinois J. Math. 1 (1957), 565–573.

Let $0 \to \mathfrak{R} \to \mathfrak{B} \xrightarrow{\sigma} \mathfrak{A} \to 0$ be an extension of the finite-dimensional (not necessarily associative) algebra $\mathfrak{A}$. Consider the following additional structure: $\mathfrak{G}$ is a finite subgroup of the group of all automorphisms and anti-automorphisms of $\mathfrak{B}$ and $\mathfrak{R}\mathfrak{G} \subset \mathfrak{R}$, so that $\mathfrak{G}$ operates naturally on $\mathfrak{A}$. $(\mathfrak{B}, \sigma)$ is then called a $\mathfrak{G}$-extension of $\mathfrak{A}$, and $\mathfrak{A}$ is $\mathfrak{G}$-segregated in $(\mathfrak{B}, \sigma)$ if $\mathfrak{R}$ has a $\mathfrak{G}$-invariant complementary subalgebra in $\mathfrak{B}$.

(I) If the order of $\mathfrak{G}$ is not a multiple of the characteristic of $\mathfrak{A}$ and if $\mathfrak{R}^2 = 0$, then the segregation of $\mathfrak{A}$ in the $\mathfrak{G}$-extension $(\mathfrak{B}, \sigma)$ implies the $\mathfrak{G}$-segregation of $\mathfrak{A}$. (II) Now suppose that, in addition, the algebras involved have ideal structures meeting certain requirements exemplified in alternative, Jordan, and Lie algebras. It follows that segregation implies $\mathfrak{G}$-segregation The additional assumptions are those required to put through the usual type of proof generalizing (I) first to nilpotent kernels and then to the general case.

The remainder of the paper deals with applications. The first is an economical proof of the Wedderburn principal theorem for Jordan algebras of characteristic $\neq 2$. The second is a refinement of the Malcev conjugacy theorem for the Wedderburn factors of a separable associative algebra $\mathfrak{A}$. If $\mathfrak{A}$ is of characteristic 0 and has an involution, and if $\mathfrak{T}$ is a self-adjoint Wedderburn factor of $\mathfrak{A}$ and $\mathfrak{S}$ any self-adjoint, semi-simple, separable subalgebra of $\mathfrak{A}$, then $\mathfrak{S}$ is orthogonally conjugate to a subalgebra of $\mathfrak{T}$. The conjugacy can be realized as the exponential of an inner derivation determined by an element of the radical. In both applications the group $\mathfrak{G}$ consists of the identity and the prescribed involution. *W. G. Lister* (Oyster Bay, N.Y.)

HOMOLOGICAL ALGEBRA

See also 4540, 4830.

4587:

Nakayama, Tadasi. On modules of trivial cohomology over a finite group. II. Finitely generated modules. Nagoya Math. J. 12 (1957), 171–176.

This is a continuation of an earlier paper [Illinois J. Math. **1** (1957), 36–43; MR **18**, 793]. Let G be a finite group, $Z[G]$ the group algebra of G over the ring Z of the rational integers. Using his 'localization' procedure with respect to Sylow subgroups and his earlier results for the case of a p-group, the author here obtains the following main result: If A is a finitely generated $Z[G]$-module without Z-torsion, then A is of trivial cohomology (if and) only if A is $Z[G]$-projective.

A second main result is the following general localization theorem. For every prime p, let Z_p denote the ring of the rational numbers that can be written with denominators not divisible by p. Let A be any $Z[G]$-module. Then, if A is of trivial cohomology, so is $A \otimes_Z Z_p$, for every prime p; conversely, if $A \otimes_Z Z_p$ is of trivial cohomology for every prime p dividing the order of G, then A is of trivial cohomology. *G. P. Hochschild* (Berkeley, Calif.)

4588:

✱Hilton, P. J. Homotopy theory of modules and duality. Symposium internacional de topología algebraica [International symposium on algebraic topology], pp. 273–281. Universidad Nacional Autónoma de México and UNESCO, Mexico City, 1958. xiv+334 pp.

An analogue to homotopy theory is developed in the category of modules over a given ring; indeed the theory is applicable in any "exact category" in the sense of Buchsbaum provided it contains "enough" injective and projective objects.

The fundamental definitions are these: A map $\varphi: A \to B$ is i-null-homotopic if it can be extended to every module A' containing A; for this it is sufficient that the condition be satisfied for any one injective A' containing A. A map $\varphi: A \to B$ is p-null-homotopic if it can be factored through every module A' of which A is a quotient; for this it is sufficient that the condition be satisfied for any one projective A' of which A is a quotient. Two maps φ, ψ are i- or p-homotopic if $\varphi - \psi$ is i- or p-null-homotopic. Notions of "homotopy-type" are immediate consequential definitions.

Let A be contained in an injective module $\bar{A}$; then the i-homotopy-type of $S(A)=\bar{A}/A$ depends on that of A only. Let $A=\underline{A}/\Omega(A)$, where $\underline{A}$ is projective; then the p-homotopy-type of $\Omega(A)$ depends on that of A only. $S(A)$, $\Omega(A)$ are analogues of the suspension and loop-space.

Using these ideas, homotopy groups and fibre-spaces of various kinds can be defined, as in topology; always, however, with the dual dichotomy of i-notions and p-notions, which coincide in topology. In particular, the "adjointness relation" $\pi(SX, Y)\approx\pi(X, \Omega Y)$ which is true in topology is not in general true in this theory.

V. Gugenheim (Baltimore, Md.)

GROUPS AND GENERALIZATIONS

See also 4501, 4513, 4514, 4587.

4589:

de Bruijn, N. G. Embedding theorems for infinite groups. Nederl. Akad. Wetensch. Proc. Ser. A. **60** = Indag. Math. **19** (1957), 560–569.

If m and n are infinite cardinals with $n\leqq m$, let Σ_m be the group of all permutations of some set M of cardinality m, and let $\Sigma_{m,n}$ be the normal subgroup consisting of all permutations σ such that the equation $\sigma(\mu)=\mu$ holds for almost all $\mu\in M$, i.e., with less than n exceptions. It is shown that: (1) Both the free product and the direct product of 2^m copies of Σ_m can be (isomorphically) embedded in Σ_m; (2) every Abelian group of order at most 2^m can be embedded in Σ_m; (3) Σ_m can be embedded in $\Sigma_m/\Sigma_{m,n}$; (4) if $k>m$ and $n=\aleph_0$, then $\Sigma_{k,n}$ cannot by embedded in Σ_m.

The first of these results is based on a more general theorem: Suppose Ω is a group generated by 2^m subgroups H_α, $\alpha\in I$, $X_{\alpha,\beta}$ is an isomorphism of H_α onto H_β, $X_{\beta,\gamma}X_{\alpha,\beta}=X_{\alpha,\gamma}$. For J a finite subset of I, let G_J be the subgroup of Ω generated by the groupH_α with $\alpha\in J$. Assume that, for any mapping φ (not necessarily one-to-one) of a finite subset J of I into I, the isomorphisms $X_{\alpha,\varphi(\alpha)}$ with $\alpha\in J$ can be jointly extended to a homomorphism of G_J onto $G_{\varphi(J)}$. Under these conditions if, for each finite set $J\subseteq I$, G_J can be embedded in S, then Ω can be embedded in S^m.

B. Jónsson (Minneapolis, Minn.)

4590:

Fuchs, L. On a directly indecomposable abelian group of power greater than continuum. Acta Math. Acad. Sci. Hungar. **8** (1957), 453–454.

This note establishes the existence of a group having the properties stated in the title. There are, in fact, 2^{2^c} non-isomorphic such groups, each having cardinality 2^c, where c is the cardinal of the continuum. The endomorphism ring of the basic example is commutative; it had been conjectured that groups of this size necessarily had non-commutative endomorphism rings.

F. B. Wright (New Orleans, La.)

4591:

Hulanicki, A. Note on a paper of de Groot. Nederl. Akad. Wetensch. Proc. Ser. A **61**=Indag. Math. **20** (1958), 114.

Using a result of de Groot [Indag. Math. **19** (1957), 137–145; MR **18**, 790] the author constructs an indecomposable Abelian group of power 2^c, with $c=2^{\aleph_0}$. The author uses a method of Bognár [Publ. Math. Debrecen **4** (1956), 509–511; MR **18**, 12]. *P. Erdős* (Haifa)

4592:

de Groot, J.; and de Vries, H. Indecomposable abelian groups with many automorphisms. Nieuw Arch. Wisk. (3) **6** (1958), 55–57.

The authors construct for any $\mathfrak{m}$ with $\aleph_0\leqq\mathfrak{m}\leqq\aleph$ two examples of directly indecomposable torsion-free abelian groups of rank $\mathfrak{m}$ having a group of automorphisms of order $2^{\mathfrak{m}}$. One of the examples is essentially the indecomposable group of rank $\mathfrak{m}$ of Bognár [Publ. Math. Debrecen **4** (1956), 509–511; MR **18**, 12]. These examples show that an indecomposable torsion-free abelian group does not necessarily contain few automorphisms as could have been supposed on the basis of earlier examples.

A. Kertész (Debrecen)

4593:

Creangă, I.; et Haimovici, Corina. Sur le sousgroupe des classes des restes premiers mod m qui satisfont la congruence $x^k\equiv1$ (mod m). An. Şti. Univ. "Al. I. Cuza" Iaşi. Secţ. I (N.S.) **3** (1957), 1–10. (Romanian. Russian and French summaries)

Several general theorems are given which refer to finite Abelian groups and relate the order of the group to the orders of the elements of the group and to the smallest common multiple of these orders. A formula is found for the number of solutions of the congruence $x^k\equiv1$ (mod m). A study is made of the case where the congruence $x^k\equiv y$ (mod m) has solutions in the classes of remainders mod m, for a given k and a variable y in the group. In this way the authors attempt to extend the arithmetic notion of radical to the classes of remainders mod m.

E. Frank (Chicago, Ill.)

4594:

Zacher, Giovanni. Sui gruppi finiti somma dei loro sottogruppi di Sylow. Rend. Sem. Mat. Univ. Padova **27** (1957), 267–275.

If G is the union of its Sylow sub-groups, if p is the least divisor of its order, and p^β the greatest power of p which divides that order; then G is solvable and its order is divisible at most by two distinct prime factors, provided it contains a proper normal sub-group with order distinct from $p^3, p^4, \cdots, p^{\beta-2}$. *H. A. Thurston* (Vancouver, B.C.)

4595:

Čunihin, S. A. On a method of obtaining subgroups and factorizations of finite groups. Dokl. Akad. Nauk SSSR **121** (1958), 243–245. (Russian)

Let $\mathfrak{G}=\mathfrak{G}_0\supset\mathfrak{G}_1\supset\cdots\supset\mathfrak{G}_\lambda=\mathfrak{E}$ be a chief series for the finite group $\mathfrak{G}$ and let $\mathfrak{J}_{\mathfrak{G}}$ be the corresponding sequence of indices $h_1, \cdots, h_\lambda$. A divisor of one of the indices is called admissible provided it is either the index itself or one or a prime-power divisor with maximal exponent. Suppose w is a nonempty subsequence of the sequence $1, 2, \cdots, \lambda$ and suppose f is a function associating to each $j\in w$ an admissible divisor f_j of the index h_j; form the product of the numbers f_j for $j\in w$. Any number h which is a product of this type is called an 'indexial' of the group $\mathfrak{G}$. For each natural number n, denote by $\Pi(n)$ the set of all prime divisors of n. The two principal theorems, which encompass results of the author [Mat. Sb. N.S. **39(81)** (1956), 465–490; **43(85)** (1957), 49–66; MR **19**, 13; **20**# 1708] and P. Hall [Proc. London Math. Soc. (3) **6** (1956), 286–304; MR **17**, 1052] pertain as follows to an arbitrary finite group $\mathfrak{G}$. If h is any indexial of $\mathfrak{G}$, then $\mathfrak{G}$ has at least one subgroup of order hc, where $\Pi(c)\subset\Pi(h)$. Given any collection of subsets $M_1, \cdots, M_\mu$ of the set $\mathfrak{J}_{\mathfrak{G}}$ (interpret $\mathfrak{J}_{\mathfrak{G}}$ as having λ members, even though there

may be fewer distinct members) such that the union of the μ subsets is $\mathfrak{J}_{\mathfrak{G}}$; then $\mathfrak{G}$ has a factorization $\mathfrak{G}=\mathfrak{M}_1\cdots\mathfrak{M}_\mu$ as a product of subgroups which are (in case $\mu>1$) pairwise permutable and such that, for each i, the order of $\mathfrak{M}_i$ is m_ic_i, where m_i is the product of the members of M_i and where $\Pi(c_i)\subset\Pi(m_i)$.

R. A. Good (College Park, Md.)

4596:

Tibiletti, Cesarina Marchionna. Sui prodotti ordinati di gruppi finiti. Boll. Un. Mat. Ital. (3) **13** (1958), 46–57.

A finite group is said to be an ordered product, that is,

$$G=A_1A_2\cdots A_s,$$

if the following conditions are satisfied: (1) the A_i are subgroups of G, $A_s(=G)$ commutes with A_{s-1}, the group $G_{s-1}=A_{s-1}A_s$ commutes with A_{s-2}, generally, the group $G_j=A_jG_{j+1}$ commutes with A_{j-1} $(2\leq j\leq s-1)$ and, finally, $G_1=A_1G_2=G$; (2) $G_j\cap A_{j-1}=\{1\}$. In most cases it is further assumed that (3) the orders of the groups A_i are coprime in pairs.

Such a product corresponds to a normal chain, if each of the groups G_i constructed in (1) is a normal subgroup of G_{i-1}, and to a principal chain, if G_i is normal in G.

The paper is concerned with the subgroup structure of an ordered product. The results established are analogues or generalizations of Sylow's theorem and are closely related to results of P. Hall [Proc. London Math. Soc. (3) **6** (1956), 286–304; MR **17**, 1052] and of G. Zappa [Boll. Un. Mat. Ital. (3) **9** (1954), 349–353; MR **16**, 793]. As an example we quote the following theorem: Let $G=A_1\cdots A_s$ be an ordered product satisfying (1) to (3); then every normal subgroup N of G is an ordered product $N=B_1B_2\cdots B_s$, where B_i is a normal subgroup of A_i (possibly the unit group). If the product for G corresponds to a principal chain, so does the product for N.

W. Ledermann (Manchester)

4597:

Dekker, Th. J. On free groups of motions without fixed points. Nederl. Akad. Wetensch. Proc. Ser. A **61**=Indag. Math. **20** (1958), 348–353.

The main result of the author's paper is the following: Let $n\geq3$. The group of motions of the Euclidean space R^n contains a free non-abelian subgroup of continuous rank without fixed points. The author states that Mycielski and Świerczkowski proved this result independently. For the literature on this question see Dekker, Indag. Math. **18** (1956), 581–595; **19** (1957), 104–107 [MR **19**, 1068].

P. Erdős (Haifa)

4598:

Baumslag, Gilbert. A theorem on infinite groups. Proc. Cambridge Philos. Soc. **53** (1957), 545–548.

The paper deals with groups G for which G/G' is a cyclic p-group. Here G' denotes the commutator group. It is well known that if, in particular, G is a finite p-group, it must be cyclic and G'/G'' is the unit-group. In generalisation of this special case, it is shown in this paper that when L is a normal subgroup of G', the order of G'/L cannot be a positive power of p and that G'/G'' is identical with the group generated by all the pth powers of its elements and, therefore, divisible by p. Moreover, G''/G''' cannot be a non-trivial cyclic p-group. Examples show that such groups exist and that G'''/G^{iv} may be a non-cyclic p-group. The divisibility of G'/G'' does not imply that G/G' is a cyclic p-group. *F. W. Levi* (Berlin)

4599:

Baumslag, Gilbert. Finite factors in infinite ascending derived series. Math. Z. **68** (1958), 465–478.

An infinite sequence of groups $H_0<H_1<H_2<\cdots$ is called an "infinite ascending derived series" if $H_{i+1}'=H_i$ for $i=0, 1, \cdots$. B. H. Neumann [Compositio Math. **13** (1956), 47–64; MR **19**, 632] has proved that no factor-group H_{i+2}/H_i can be finitely generated, but he has shown by an example that there may be some H_{j+1}/H_j of finite order. In this paper it is proved that there may be even an infinite number of such factor-groups of finite order. Given an arbitrary sequence of odd integral numbers $n_0, n_1, \cdots, n_k, \cdots$ satisfying $(n_k, n_{k+1})=1$ for $k=0, 1, 2, \cdots$, the author gives a construction of an infinite ascending derived series where $|H_{2k+1}/H_{2k}|=n_k$, whereas the orders of H_{2k}/H_{2k-1} are all infinite and of the same power. The orders of the finite factor-groups may therefore be bounded. Without proof it is stated that, by a similar but more complicated construction, the condition that the n_k be odd may be dispensed with. The construction of the series is based on groups $B(m, n)$, $(m, n)=1$, $|B(m, n)|=m^n$, $B'(m, n)$ being cyclic of order m and coincident with its centre. By a constructional procedure which generalises B. H. Neumann's method of "crown product", the infinite ascending derived series with the required properties is obtained. *F. W. Levi* (Berlin)

4600:

Lambek, Joachim. Goursat's theorem and the Zassenhaus lemma. Canad. J. Math. **10** (1958), 45–56.

Let $\rho=(R, A, B)$ be a binary relation between two groups A, B $(R\subset A\times B)$ and $\rho^-=(R^-, B, A)$ the converse of ρ. Following Riguet [Bull. Math. Soc. France **76** (1948), 114–155; MR **10**, 502], ρ is called difunctional if and only if $\rho\rho^-\rho\leq\rho$. The binary relation ρ is defined to be homomorphic if and only if its graph R is a subgroup of the direct product $A\times B$. If ρ is homomorphic then it is difunctional. In this case one easily verifies that for any subgroup A' of A the set $A'\rho=\{b|a\rho b \text{ for some } a\in A'\}$ forms a subgroup of B. In particular, $A\rho$ is called the range of ρ. A homomorphic equivalence relation is a congruence relation, a homomorphic relation which is transitive and symmetric without necessarily being reflexive is called subcongruence. Any subcongruence $\kappa=(K, A, A)$ on A induces a congruence relation $(K, A\kappa, A\kappa)$ on $A\kappa$. The factor group of $A\kappa$ modulo κ, called a subfactor of A, is written $A\kappa/\kappa$. Of the results established in the present paper we point out the following (partially due to Riguet), which gives Goursat's characterization of the subgroups of the direct product of two groups: If $\rho=(R, A, B)$ is a homomorphic relation (between the groups A, B) then $\kappa=\rho\rho^-$ is a subcongruence of A with range $B\rho^-$, $\lambda=\rho^-\rho$ a subcongruence of B with range $A\rho$, and ρ induces an isomorphism μ between subfactors $A\kappa/\kappa$ and $B\lambda/\lambda$ such that $(a\kappa)\mu(b\lambda)$ if and only if $a\rho b$. Conversely, every isomorphism between subfactors of A and B is induced in this manner. — A like result is shown for a general class of algebras, including loops and quasigroups, and it is used to obtain general forms of the Zassenhaus lemma and the Jordan-Hölder-Schreier theorem. *O. Borůvka* (Brno)

4601:

Vasil'ev, A. M. On the orthogonal subgroups of classical compact Lie groups. Dokl. Akad. Nauk SSSR **121** (1958), 18–21. (Russian)

Let $E(N)$ denote an N-dimensional Euclidean space. Then the classical compact groups $\mathrm{O}(n)$, $\mathrm{U}(n)$, $\mathrm{Sp}(2n)$ act

in a natural manner on $E(n)$, $E(2n)$, $E(4n)$, respectively. The resulting representations will be called R-representations of these groups. A group G consisting of orthogonal transformations of $E(N)$ is said to be of type ρ if G is a direct product of groups $O(l_a)$, $U(m_k)$, $Sp(2n_r)$ which act independently as R-representations on mutually orthogonal subspaces $E(l_a)$, $E(2m_k)$, $E(4n_r)$, while leaving invariant every point of the maximal subspace which is orthogonal to all of $E(l_a)$, $E(2m_k)$, $E(4n_r)$. The author announces, among other things, the following theorem: The intersection $G_1 \cap G_2$ of two groups of type ρ of $E(N)$ is a direct product of groups $O(l_\alpha)$, $U(m_\chi)$, $Sp(2n_\rho)$ which act independently on mutually orthogonal subspaces $E(l_\alpha\lambda_\alpha)$, $E(2m_\chi\mu_\chi)$, $E(4n_\rho\nu_\rho)$ as λ_α-, μ_χ-, ν_ρ-fold R-representations, respectively, and leave invariant every point of the maximal subspace $E(N_0)$ which is orthogonal to all of $E(l_\alpha\lambda_\alpha)$, $E(2m_\chi\mu_\chi)$, $E(4n_\rho\nu_\rho)$. The totality of orthogonal transformations of $E(N)$ which commute with every element of $G_1 \cap G_2$ is the direct product of the groups $O(\lambda_\alpha)$, $U(\mu_\chi)$, $Sp(2\nu_\rho)$, $O(N_0)$ which act independently on the same system of subspaces as l_α-, m_χ-, n_ρ-, 1-fold R-representations, respectively. *R. Ree* (Vancouver, B.C.)

4602:

Kimura, Naoki. Note on idempotent semigroups. I. Proc. Japan Acad. 33 (1957), 642–645.

D. McLean [Amer. Math. Monthly **61** (1954), 110–113; MR **15**, 681] proved: To each idempotent semigroup S there exist, up to isomorphism, a unique semilattice Γ and a disjoint family of so-called rectangular subsemigroups of S indexed by Γ, $\{S_\gamma : \gamma \in \Gamma\}$, such that $S = \bigcup\{S_\gamma : \gamma \in \Gamma\}$ and $S_\alpha S_\beta \subset S_{\alpha\beta}$ for all $\alpha, \beta \in \Gamma$. (Here S_γ is called rectangular if $aba = a$ for every $a, b \in S_\gamma$.)

Let A, B be idempotent semigroups both of which have the same structure semilattice Γ, and $\varphi: A \to \Gamma$, $\psi: B \to \Gamma$ be their canonical homomorphisms. Then $P = \{(x, y) | x \in A, y \in B, \varphi(x) = \psi(y)\}$ (considered as a subsemigroup of the direct product $A \times B$) forms a semigroup which is called the spined product of A and B with respect to Γ.

An idempotent semigroup S is called left regular, resp. right regular, resp. regular if it satisfies the identity $aba = ab$, resp. $aba = ba$, resp. $abaca = abca$. The author proves: An idempotent semigroup is regular if and only if it is a spined product of a left regular idempotent semigroup and a right regular idempotent semigroup.

Št. Schwarz (Bratislava)

4603:

Yamada, Miyuki; and Kimura, Naoki. Note on idempotent semigroups. II. Proc. Japan Acad. 34 (1958), 110–112.

An idempotent semigroup S is called left normal, resp. right normal, resp. normal if it satisfies the identity $abc = acb$, resp. $bca = cba$, resp. $abca = acba$.

Let $S = \bigcup\{S_\gamma; \gamma \in \Gamma\}$ be the decomposition of S in the sense of the paper reviewed above. The purpose of this note is to find necessary and sufficient conditions for S to be left normal, resp. right normal, resp. normal. This is done by means of families of functions $\phi_\beta{}^\alpha : S_\alpha \to S_\beta$ for $\alpha > \beta$ and by means of the notion of the spined product (introduced in the foregoing paper).

Št. Schwarz (Bratislava)

4604:

Kimura, Naoki. Note on idempotent semigroups. III, IV. Proc. Japan Acad. 34 (1958), 113–114, 121–123.

Part III. Two equivalence relations P and Q on an idempotent semigroup S are introduced: 1) $xPy \Leftrightarrow xy = y$ and $yx = x$. 2) $xQy \Leftrightarrow xy = x$ and $yx = y$. The theorems give conditions for situations of the following sort: a) P constitutes a congruence on S, b) P is a congruence and the quotient semigroup S/P is left normal (in the sense of the foregoing note), c) P and Q are congruences and S/P is left normal and S/Q is right normal. Proofs are not given.

Part IV. Any identity of (at most) three variables on idempotent semigroups is equivalent (in a well-defined sense) to one of 18 explicitly given properties or identities. Such identities are, for instance: triviality, $x = y$; commutativity, $xy = yx$; left normality, $xyz = xzy$; rectangularity, $xyx = x$; and so on. Each of them (excluding one) leads to special idempotent semigroups studied by the author in previous papers [see Parts I, II, III reviewed above]. Proofs are omitted. *Št. Schwarz* (Bratislava)

4605:

Vakselj, Anton. Faktorhalbgruppen. Glasnik Mat.-Fiz. Astr. Društvo Mat. Fiz. Hrvatske ser. II. **12** (1957), 9–16. (Serbo-Croatian summary)

The author shows, in the usual way, that the multigroup determined by the left cosets of a subgroup of a finite group can be interpreted as a partial subgroupoid of a groupoid. *G. B. Preston* (Shrivenham)

TOPOLOGICAL GROUPS AND LIE THEORY

See also 4774, 4789, 4821.

4606:

Hulanicki, A. On cardinal numbers related with locally compact groups. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **6** (1958), 67–70.

Let G be a locally compact group and $\theta(G)$ the minimal cardinality of a family of open sets having as intersection the unit of G. Then the cardinality of G is greater than or equal to $2^{\theta(G)}$. Furthermore, if G is the union of $2^{\theta(G)}$ compact subsets, then the cardinality of G is equal to $2^{\theta(G)}$. *K. deLeeuw* (Stanford, Calif.)

4607:

Anderson, R. D. The algebraic simplicity of certain groups of homeomorphisms. Amer. J. Math. **80** (1958), 955–963.

Let G be a group acting on a Hausdorff space as a transformation group. The main theorem asserts that if the action possesses GK-transitivity — defined by its effect on a collection K of closed sets — then G is simple and if h is an element of G different from the neutral element, every element of G is the product of six conjugates of h and h^{-1}. Cases covered by the theorem: the group of orientation preserving self-homeomorphisms of the 2-sphere, the 3-sphere, the universal plane curve; the group of self-homeomorphisms of the Cantor set, the universal curve, the space of rational numbers, the space of irrational numbers. *P. A. Smith* (New York, N.Y.)

4608:

Dieudonné, Jean. Groupes de Lie et hyperalgèbres de Lie sur un corps de caractéristique $p > 0$. VII. Math. Ann. **134** (1957), 114–133.

[See the previous papers in this series: Comment. Math. Helv. **28** (1954), 87–118; Amer. J. Math. **77** (1955), 218–244; Math. Z. **63** (1955), 53–75; Amer. J. Math. **77** (1955), 429–452; Bull. Soc. Math. France **84** (1956), 207–239; Amer. J. Math. **79** (1957), 331–388; MR **16**, 12, 789; **17**,

174; **20** #930, #931.] The author gives a complete classification, up to isogeny [cf. part VI], of formal abelian groups G of finite dimension over an algebraically closed field K of characteristic $p\neq 0$. G, K will be understood to have these meanings throughout the present review. The fundamental theorems are as follows. (1) G is isogeneous to a direct product of simple groups and Witt groups W_m [cf. part II]. (2) A simple group G is isogeneous to one of the groups $G_{n,0,m}$ [cf. part III], where m and n are coprime. (3) The ring of endomorphisms of the simple group $G_{n,0,m}$ is isomorphic to an order in the central division algebra of rank $(m+n)^2$ and invariant $n/(m+n)$ over the field of p-adic numbers [cf. M. Deuring, Algebren, Springer, Berlin, 1937; pp. 112–113].

In part IV the author associated with G a matrix $M(G)$ whose coefficients lie in a certain non-commutative ring $\mathfrak{E}^+$; moreover, G_1 and G_2 are isomorphic if, and only if, $M(G_1)$ and $M(G_2)$ are equivalent over $\mathfrak{E}^+$. It is proved in the present paper that G_1 and G_2 are isogeneous if, and only if, $M(G_1)$ and $M(G_2)$ are equivalent over a certain ring $\mathfrak{A}$ which contains $\mathfrak{E}^+$. ($\mathfrak{A}$, $\mathfrak{E}^+$ are defined as follows. Let W denote the ring of Witt vectors $A=(a_0, a_1, \cdots)$ over K; W is a complete valuation ring, $\pi=(0, 1, 0, \cdots)$ a uniformizing parameter, and the mapping $A\to A^\sigma=(a_0{}^p, a_1{}^p, \cdots)$ an automorphism of W. Then $\mathfrak{E}^+$ is the set of all formal non-commutative integral power series $A_0+tA_1+t^2A_2+\cdots$, where the $A_i\in W$ and the indeterminate t satisfies $At=tA^\sigma$ for all $A\in W$. $\mathfrak{A}$ is the set of all power series $t^hA_h+t^{h+1}A_{h+1}+\cdots$ ($h=0, \pm 1, \pm 2, \cdots$).)

It is proved that (i) $\mathfrak{A}$ has a Euclidean division algorithm and (ii) the $\mathfrak{A}$-modules which intervene are all bounded (i.e., they have non-zero annihilator). (i) enables the powerful theory of modules over principal ideal rings to be applied and (ii) allows a further reduction to problems of similarity, etc., within $\mathfrak{A}$ itself [cf. N. Jacobson, The theory of rings, Amer. Math. Soc., New York, 1943; MR **5**, 31]. The author also shows that each $u\in\mathfrak{A}$ can be written in 'quasi-normal' form $u=\pi^{\alpha_0}t^{\beta_0}u_0+\cdots+\pi^{\alpha_r}t^{\beta_r}u_r$, where the u_i are invertible elements of $\mathfrak{E}^+$, the α_i form a strictly decreasing sequence of integers $\geqq 0$, and the β_i a strictly increasing sequence of integers.

The above holds, in fact, whenever K is perfect; the algebraic closure of K is required in proving the following very explicit reducibility criterion: let $u=\pi^nu_0+\pi^{\alpha_1}t^{\beta_1}u_1+\cdots+\pi^{\alpha_{k-1}}t^{\beta_{k-1}}u_{k-1}+t^mu_k$ be an element of $\mathfrak{E}^+$ not divisible by π, expressed in quasi-normal form; then u is irreducible in $\mathfrak{A}$ if, and only if, m and n are coprime and $\alpha_i/n+\beta_i/m>1$ for $1\leqq i\leqq k-1$; u, when irreducible, is similar to π^n-t^m. The fundamental theorems are deduced fairly easily from this criterion and the general theory of equivalence over principal ideal rings.

Finally, the author constructs an infinity of formal abelian groups of dimension 2, any two of which are isogeneous but no two of which are isomorphic. This (and other evidence) suggests that the classification of abelian groups up to isomorphism may be difficult.

G. E. Wall (Sydney)

4609:

Dieudonné, Jean. Lie groups and Lie hyperalgebras over a field of characteristic $p>0$. VIII. Amer. J. Math. **80** (1958), 740–772.

In the present paper the author initiates the study of group extensions. (Unless otherwise stated, 'group' will always mean 'formal Lie group of finite dimension over an algebraically closed field K of characteristic $p\neq 0$'.) The basic concepts are the factor group H/N and natural homomorphism h: $H\to H/N$ corresponding to a normal 'typical' subgroup N of the group H; the definition given here is more direct than that in part VI and avoids the hyperalgebra. A sequence of groups and homomorphisms $0\to A\xrightarrow{u}P\xrightarrow{v}G\to 0$ is called exact if it can be carried by isomorphisms into a sequence $0\to N\xrightarrow{j}H\xrightarrow{h}H/N\to 0$ (N typical, j the natural injection), and such an exact sequence defines an extension (P, u, v) of G by A. Equivalence of extensions is defined in the usual way.

In non-formal group theory a splitting extension of $\mathfrak{G}$ by $\mathfrak{A}$ gives rise to a 'semi-direct' product $\mathfrak{G}\mathfrak{A}$; once the mode of action of $\mathfrak{G}$ on $\mathfrak{A}$ has been fixed, the semi-direct product can be defined abstractly as the set of pairs (g, a) ($g\in\mathfrak{G}$, $a\in\mathfrak{A}$) under a certain law of multiplication. The author gives an analogous abstract definition of a semi-direct product of formal groups G, A, and the corresponding extension of G by A is called trivial. It is pointed out that (P, u, v) is not necessarily trivial when P contains a subgroup G' isomorphic to G and such that $u(A)\wedge G'=e$, $u(A)\vee G'=P$ ($\wedge$, $\vee$ are defined in part VI); these conditions ensure merely that P is isogeneous to a semi-direct product of G and A. When G acts trivially on A, the semi-direct product becomes the direct product $G\times A$. A group H is called quasi-direct product of its normal subgroups M, N if $M\wedge N=e$, $M\vee N=H$; H is then isogeneous, but not necessarily isomorphic, to $M\times N$ and the corresponding extension of M by N is called quasi-trivial.

The main result of this paper is theorem A: Every solvable group is isogeneous to a direct product $P=D\times Q$, where D is a divisible abelian group and Q a semi-direct product of the largest unipotent subgroup of P and a torus. Theorem A follows readily from results in VI and theorem B: If P is a solvable group and D the largest divisible abelian group in the center of P, then D is a quasi-direct factor of P.

For the proof of theorem B, the author introduces formal G-modules and their (non-formal) cohomology groups. Let G be a group, A an abelian group, with respective laws of composition $\mathbf{st}=(\phi_i(\mathbf{s}, \mathbf{t}))_{i\in I}$, $\mathbf{x}+\mathbf{y}=(\psi_j(\mathbf{x}, \mathbf{y}))_{j\in J}$. Then A becomes a G-module if we prescribe a system of power series $\mathbf{s}\cdot\mathbf{x}=(f_j(\mathbf{s}, \mathbf{x}))_{j\in J}$, without constant terms, such that: (i) $\mathbf{s}\cdot(\mathbf{x}+\mathbf{y})=\mathbf{s}\cdot\mathbf{x}+\mathbf{s}\cdot\mathbf{y}$; (ii) $\mathbf{e}\cdot\mathbf{x}=\mathbf{x}$; (iii) $\mathbf{s}\cdot(\mathbf{t}\cdot\mathbf{x})=(\mathbf{st})\cdot\mathbf{x}$. An m-cochain is defined as a system of power series $\mathbf{g}(\mathbf{s}_1, \cdots, \mathbf{s}_m)=(g_j(\mathbf{s}_1, \cdots, \mathbf{s}_m))_{j\in J}$, without constant terms, in m independent sets $\mathbf{s}_h=(s_{hi})_{i\in I}$ of variables ($1\leqq h\leqq m$). Sum of cochains, coboundary, cohomology groups $H^m(G, A)$ are now defined in an evident way. A difficulty arises, however, in defining 1-coboundaries because there are no 'elements' in A; $H^1(G, A)$ is identified with the group of homomorphisms of G into A when G acts trivially on A and is left undefined otherwise. The usual correspondence between elements of $H^2(G, A)$ and classes of equivalent extensions of G by A holds good, and the zero of $H^2(G, A)$ corresponds to the trivial extensions as defined above. The principal tool in proving theorem B is the lemma: Suppose that the group G and its subgroups and factor groups act trivially on the abelian group A and let H be a normal subgroup of G such that $H^1(H_L, A_L)=0$ for any perfect extension L of the field K (H_L, A_L are the groups over L defined by the same systems of power series as H, A); then there exists an exact sequence $0\to H^2(G/H, A)\to H^2(G, A)\to H^2(H, A)$. The proof is not easy and depends on the construction of an auxiliary G-module of infinite dimension. To complete the proof of theorem B, the

author shows, using part VI, that there is a sequence of subgroups of the solvable group $P: D\subset R_1\subset R_2\subset\cdots\subset R_m\subset P$, each normal in its successor, such that R_1 is the center of P, R_{i+1}/R_i is isomorphic to the additive group I_∞ of K for $1\leqq i\leqq m-1$, and P/R_m is a torus T. After showing independently that all central extensions of I_∞ or T by D are quasi-trivial, he applies a modified form of the lemma to obtain the theorem by induction.

As an application, all 2-dimensional groups are determined up to isogeny. The 2-dimensional nilpotent groups G, with laws of composition of the form

$$\phi_1(x, y; x', y')=x+x',$$
$$\phi_2(x, y; x', y')=y+y'+f(x, x'),$$

provide interesting examples. Firstly, if $f(x, x')=\sum_{h>k\geq 0} e_{hk}x^{p^h}x'^{p^k}$ and the e_{hk} are algebraically independent over the prime field F_p of K, then G is a noncommutative unipotent group which is not representable (i.e., not isogeneous to a subgroup of a full linear group). Secondly, if $f(x, x')=\sum_{h\geq 1} e_h x^{p^h}x'$ and the e_h are algebraically independent over F_p, then G is isogeneous to a subgroup of GL(3) but not to any algebraic group.

G. E. Wall (Sydney)

TOPOLOGICAL ALGEBRA

4610:

Nachbin, Leopoldo. On the operational calculus with differentiable functions. Proc. Nat. Acad. Sci. U.S.A. **44** (1958), 698–700.

Denote by $C^n(R)$ the algebra of all real functions which are defined on the real line R and have continuous derivatives up to order n. Under the topology determined by uniform convergence on compact sets of the functions along with their n derivatives, $C^n(R)$ is a topological algebra. It contains as a subalgebra the collection $P(R)$ of all polynomials with real coefficients. A given topological algebra A is said to have an operational calculus with $C^n(R)$ if there exists a continuous mapping of $C^n(R)\times A$ into A such that (p, x) maps into $p(x)$ for arbitrary $p\in P(R)$ and $x\in A$. If $(p, x)\rightarrow p(x)$ is a continuous mapping of $P(R)\times A$ into A (where $P(R)$ is given the topology induced on it by the topology of $C^n(R)$), then A is said to have a pre-operational calculus with $C^n(R)$. The author announces a theorem which gives a necessary and sufficient condition for certain special classes of topological algebras to have a pre-operational calculus with $C^n(R)$. *C. E. Rickart* (New Haven, Conn.)

FUNCTIONS OF REAL VARIABLES

See also 4569, 4692, 4877.

4611:

Popa, Ilie. Une condition suffisante de non-mesurabilité pour les fonctions bornées. Rev. Univ. "Al. I. Cuza" Inst. Politehn. Iaşi **1** (1954), 6–8. (Romanian. Russian and French summaries)

On démontre la proposition: une condition suffisante afin qu'une fonction $f(x)$ bornée dans l'intervalle (a, b) soit non-mesurable est qu'une expression telle que

$$\varphi(x)=\sum_{\nu=0}^{n-1}\frac{\xi_{\nu+1}-\xi_\nu}{b-a}f\left[\frac{\xi_{\nu+1}-\xi_\nu}{b-a}x+\frac{b\xi_\nu-a\xi_{\nu+1}}{b-a}\right]-f(x)$$

($\xi_0=a, \xi_1, \cdots, \xi_{n-1}, \xi_n=b$, points de l'intervalle (a, b)) garde un signe constant presque partout dans l'intervalle. Conséquence: Si la fonction $f(x)$ est continue dans l'intervalle (a, b), l'équation $\varphi(x)=0$ a toujours une racine dans le même intervalle. Cas particuliers: On a les mêmes résultats en prenant pour $\varphi(x)$

$$\theta f[\theta x+(1-\theta)a]+(1-\theta)f[(1-\theta)x+\theta b]-f(x),\ 0\leqq\theta\leqq 1,$$

ou

$$f(\tfrac{1}{2}(x+a))+f(\tfrac{1}{2}(x+b))-2f(x).$$

Résumé de l'auteur

4612:

Ostmann, Hans-Heinrich. Eine Bemerkung über quasiwachsende Funktionen. Math. Nachr. **18** (1958), 127–128.

The author strengthens a theorem of N. I. Achieser [Lectures on the theory of approximation, OGIZ, Moscow-Leningrad, 1947; German edition by Akademie-Verlag, Berlin, 1953; MR **10**, 33; **15**, 867; p. 235] by imposing a condition upon the midpoints of the intervals occurring there. This is a consequence of the following theorem proved by the author: If infinitely many intervals $[a_\nu-l, a_\nu+l]$ are given and ρ is an arbitrary positive number, then there are infinitely many partial intervals of equal length

$$[b_{\nu_\kappa}-l', b_{\nu_\kappa}+l']\subset[a_{\nu_\kappa}-l, a_{\nu_\kappa}+l]\ (\kappa=1, 2, \cdots;\ l'=l'(\rho)>0)$$

whose centers b_{ν_κ} are congruent to each other mod $+\rho$. *A. Rosenthal* (Lafayette, Ind.)

4613:

Mibu, Yoshimichi. On quasi-continuous mappings defined on a product space. Proc. Japan Acad. **34** (1958), 189–192.

H. Hahn [Math. Z. **4** (1919), 306–313] proved the theorem: If $f(x, y)$, for every fixed x, is a continuous function of y and, for every fixed y, is a continuous function of x, then the set of continuity points of f is dense in the plane. The author generalizes Hahn's result in the following manner. Theorem 1: Let X and Y be two topological spaces and M a metric space. Let $f(x, y)$ be a mapping of the product space $X\times Y$ into M. Assume that the following conditions are satisfied: 1) For every fixed $x\in X$, the mapping $f(x, y)$ is a continuous mapping of Y into M; 2) there exists a set H which is dense in Y such that $f(x, y)$ is a continuous mapping of X into M for every fixed $y\in H$; 3) every open subset of X is of the second category; 4) Y satisfies the first axiom of countability. Then $f(x, y)$ is a "quasi-continuous" mapping of $X\times Y$ into M, which means that the set of discontinuity points of f is of the first category. Theorem 2 is analogous, except that in 4) Y is to be replaced by X and instead of 2) and 3) we assume: 2') there exists a subset L of Y which is of the first category such that $f(x, y)$ is a "quasi-continuous" mapping of X into M for every fixed $y\in Y-L$; 3') every open subset of X and Y is of the second category. *A. Rosenthal* (Lafayette, Ind.)

4614:

Labutin, D. N. On the mean rate of change of functions. Kabardin. Gos. Ped. Inst. Uč. Zap. **12** (1957), 65–71. (Russian)

The author shows that if an interval Δx is divided into n equal parts Δx_i, then the average rate of change of a

continuous function for the whole interval is the average of the average rates of change over the pieces:

$$\frac{\Delta y}{\Delta x}=n^{-1}\sum_{i=0}^{n-1}\frac{\Delta y_i}{\Delta x_i}.$$

R. P. Boas, Jr. (Evanston, Ill.)

4615:

Popov, B. S. Sur quelques intégrales définies. Fac. Philos. Univ. Skopje. Sect. Sci. Nat. Annuaire **9** (1956), 15–20. (Serbo-Croatian summary)

Le but de la Note présente est de donner la démonstration de la formule

$$\int_0^1 x^{r-1}(1-x^l)^{q-1}(1-zx^p)^{-\alpha}dx=\frac{1}{l}\sum_{i=0}^{\infty}\frac{(\alpha)_i}{i!}B\left(\frac{r+ip}{l},q\right)z^i,$$

où p, q, l, r sont des nombres positifs, B l'intégrale eulérienne de première espèce et

$$(\alpha)_i=\alpha(\alpha+1)\cdots(\alpha+i-1),$$
$$(\alpha)_0=0.$$

Résumé de l'auteur

4616:

Marcus, S. Sur un théorème de M. S. Stoïlow, concernant les fonctions continues d'une variable réelle. Rev. Math. Pures Appl. **2** (1957), 409–412.

f: fonction réelle définie et continue sur un intervalle $I=[a, b]$ de l'axe réel X. J: ensemble (intervalle) des valeurs prises par f. F: ensemble des $x\in I$ où f possède une dérivée unique et finie. Un théorème de M. S. Stoïlow [Bull. Soc. Math. France **53** (1925), 135–148; C. R. Acad. Sci. Paris **179** (1924), 807–810, 1585–1586] affirme entre autres: Il existe un ensemble numérique $Z\subseteq J$ de mesure (lebesguienne) nulle ("ensemble exceptionnel vertical") tel qu'en chaque point isolé de "l'ensemble de niveau" $\{x; f(x)=\xi\}$ où $\xi\in J-Z$ la dérivée (bilatérale) existe. Ce résultat est utilisé pour établir simplement et généraliser des théorèmes de "type vertical" dus à S. Saks, S. Banach, et S. Minakshisundaram. Voici l'un d'eux: Si l'image $f(A)$ de tout ensemble $A\subseteq I$ de mesure nulle est aussi de mesure nulle (propriété (N) de Lusin), alors $f(I-F)$ est de mesure nulle. {Remarque du rapporteur: Un théorème "clef" en théorie géométrique de la dérivation se trouve chez G. Choquet et Chr. Pauc, Bull. Sci. Math. (2) **70** (1946), 12–21 [MR **8**, 257].}

Chr. Pauc (Nantes)

4617:

Chen, Yung-Ming. On a maximal theorem of Hardy and Littlewood and theorems concerning Fourier constants. Math. Z. **69** (1958), 418–422.

If f is a non-negative integrable function on the interval $a\leqq x\leqq b$, let $\Theta(x)=\Theta(x; f)=\sup(x-\xi)^{-1}\int_\xi^x f(t)dt$, $a\leqq\xi\leqq x$. Let Φ be a function such that for some $\delta>0$, $\Phi(x)x^{-1-\delta}$ is increasing on $x>0$, and for some $k>1$, $\Phi(x)x^{-k}$ is decreasing. Let Ψ be a positive non-decreasing function such that for some $\delta'>0$, $\Psi(x)x^{-1+\delta'}$ is decreasing on $(0, a)$.

The main theorem of the paper asserts that there is a constant K depending only on Φ such that

$$\int_a^b \Phi\{\Theta(x)\}dx\leqq K\int_a^b \Phi\{f(x)\}dx,$$

which, for the special function $\Phi(x)=x^p$, $p>1$, is a well-known and important inequality of Hardy and Littlewood [Acta Math. **54** (1930), 81–116]. The proof is based on a lemma which states that there is a constant K, again depending on Φ only, such that

$$\int_0^a \frac{\Phi\{F(x)/x\}}{\Psi(x)}dx\leqq K\int_0^a \frac{\Phi\{f(x)\}}{\Psi(x)}dx,$$

where $F(x)=\int_0^x f(t)dt$. Applications similar to those made originally by Hardy and Littlewood are given.

In addition, some applications are given to theorems concerning Fourier constants. The following is an example. Let f be positive and even in $(-\pi, \pi)$ and non-increasing in $(0, \pi)$, and let $\{a_n\}$ be the sequence of Fourier cosine coefficients of f. Then $\Phi\{f(x)\}/\Psi(x)$ is integrable on $(0, \pi)$ if and only if

$$\sum_1^\infty \frac{\Phi\{n|a_n|\}}{n^2\Psi(1/a_n)}<\infty.$$

K. T. Smith (Madison, Wis.)

4618:

Natanson, I. P. On an extremum problem concerning increasing polynomials. Vestnik Leningrad. Univ. **13** (1958), no. 7, 103–108. (Russian. English summary)

The author solves the problem $I=\int_a^b\varphi(x)P(x)dx=\min.$, where $\varphi(x)\geqq 0$, and $P(x)$ is a polynomial of degree $\leqq n$ with one, two or three fixed leading coefficients, which is positive on $[a, b]$ and increasing on $(-\infty, +\infty)$. Assuming $P(a)=0$, one has $I=\int_a^b wP'dx$, $w(x)=\int_x^b\varphi(t)dt$. Putting $P'=M^2+N^2$ and writing the expansions of M, N with respect to orthogonal polynomials with weight w, the author can find the coefficients of the expansions.

G. G. Lorentz (Syracuse, N.Y.)

4619:

Marusciac, I. Sur le problème no. 5383 de la Gazeta Matematică de l'année 1940. Gaz. Mat. Fiz. Ser. A (N.S.) **10(63)** (1958), 463–467. (Romanian. French and Russian summaries)

Dans cette note on détermine toutes les polynômes $P(x)$ qui vérifient la relation

$$P(x-a)P(x-b)=P(x)Q(x),$$

pour toutes les nombres réels a et b. Le problème s'étend aussi pour le cas quand a et b sont complexes. On détermine aussi le polynôme de grade minime et on montre que ce polynôme est unique, abstraction fait d'une transformation linéaire.

Résumé de l'auteur

4620:

Soble, A. B. Majorants of polynomial derivatives. Amer. Math. Monthly **64** (1957), 639–643.

Let $P(x)=\sum_{j=0}^n c_jx^{n-j}$, $c_i>0$. Then $P'/P\leqq nx^{-1}$, $x>0$, and $P'/P\leqq(2x)^{-1}n$, $0<x\leqq\min(c_j/c_{j-1})$. Let $c_0=1$, $c=1+\max|c_j|$. Then $|P'/P|\leqq n(x-c/2)^2\cdot x^{-1}(x-c)^{-2}$, $x>c$. Finally, let $c_j>0$, $c_j\uparrow$. Then $P'/P\leqq(n+1)(kx)^{-1}$, $0<x\leqq\exp(-k/e)$, $k\geqq e$.

G. Szegö (Stanford, Calif.)

4621:

Constantinesco, Florent. Sur un théorème de Marcel Riesz. C. R. Acad. Sci. Paris **247** (1958), 256–257.

The theorem of the title is a special case of the following new one: For real polynomials P and Q of degrees differing by at most 1, whose zeros are all real and interlaced, define $\delta(P, Q)$ as the minimum distance of a zero of Q from a zero of P; then $\delta(P', Q')\geqq\delta(P, Q)$.

C. Davis (Providence, R.I.)

MEASURE AND INTEGRATION

See also 4792, 4996, 5018.

4622:

Tortrat, Albert. Mesures singulières et automorphismes sur un pavé E_r. Publ. Sci. Univ. Alger. Sér. A **3** (1956), 173–180.

(I) If P is a nowhere dense perfect set interior to E_r

(=unit cube in r-space), and $\alpha>0$, then $m(hP)>1-\alpha$ for some automorphism h of E_r. (II) If the union X of a sequence of perfect sets is dense in E_r, then $m(hX)=1$ for some automorphism h of E_r. Theorem II was obtained by the reviewer and Ulam [Ann. of Math. (2) **42** (1941), 874–920; MR 3, 211] as a corollary of a characterization of measures topologically equivalent to Lebesgue measure (and theorem I can be obtained in the same way). The present paper gives direct proofs. Actually, both theorems should be attributed to L. E. J. Brouwer, who proved them in the case $r=2$ [Math. Ann. **79** (1919), 212–222]. Finally, it is shown that a set X with $m(X)=0$ can support a singular measure of Lebesgue-Stieltjes type if and only if X contains a perfect set.

J. C. Oxtoby (Bryn Mawr, Pa.)

4623:

Rényi, A. On mixing sequences of sets. Acta Math. Acad. Sci. Hungar. **9** (1958), 215–228.

A sequence of measurable sets $\{A_n\}$ in a measure space is called by the author a strongly mixing sequence if $\lim_{n\to\infty} \mu(A_nB)=\alpha\mu(B)$ for any measurable set B satisfying $\mu(B)<\infty$. If the measure space is a probability space $[\Omega, A, P]$ the defining relation for strong mixing becomes, with the usual notation for conditional probability, $\lim_{n\to\infty} P(A_n|B)=\alpha$.

The author proves several theorems showing that the property of strong mixing of a sequence of sets depends only on the internal relation of the sets in the sequence and is independent of the underlying probability measure. This independence is also enjoyed, under certain conditions, by limiting distributions of sequences of random variables (theorem 4). More precisely, theorems 1 and 2 state: A necessary and sufficient condition that a sequence $\{A_n\}$ of measurable sets be a strongly mixing sequence is that $\lim_{n\to\infty} P(A_n|A_k)=\alpha$ for $k=0, 1, 2, \cdots$. In this case $\lim Q(A_n|B)=\alpha$ for every probability measure Q equivalent to P.

Similar theorems are proved for weak mixing and several applications are made. *Y. N. Dowker* (London)

4624:

Beck, Anatole; and Schwartz, J. T. A vector-valued random ergodic theorem. Proc. Amer. Math. Soc. **8** (1957), 1049–1059.

An individual ergodic theorem (generalizing an unpublished result of Kakutani) for a certain type of vector-valued function is proved and used to obtain a generalization of the law of large numbers. Let $\mathfrak{X}$ be a reflexive B-space; (S, Σ, m) a σ-finite measure space; $L_1(S, \mathfrak{X})$ the $\mathfrak{X}$-valued integrable functions on S; T_s, $s\in S$, a strongly measurable function mapping S into the family of bounded linear operators in $\mathfrak{X}$; and h a measure-preserving transformation in S. Then, for every X in $L_1(S, \mathfrak{X})$, the strong limit $\lim n^{-1}\sum_{i=1}^{n} T_sT_{h(s)}\cdots T_{h^{i-1}(s)}X(h^i(s))$ exists almost everywhere on S and defines a function in $L_1(S, \mathfrak{X})$. If $m(S)<\infty$ this limit also exists in $L_1(S, \mathfrak{X})$. A number of interesting corollaries and related material are given. We mention only the following theorem. Let $\{X_i\}$ be an independent sequence of $\mathfrak{X}$-valued random variables having the same distribution. Then for any linear operator T in $\mathfrak{X}$ with $|T|\leqq 1$ the limit $\lim n^{-1}\sum_{i=1}^{n} T^iX_i$ exists almost surely. *N. Dunford* (New Haven, Conn.)

4625:

Dombrowski, Peter. Zur Substitutionsregel für n-dim. Lebesguesche Integrale in reellen Zahlräumen und Mannigfaltigkeiten. Bonn. Math. Schr. no. **1** (1957), 98 pp.

Das Hauptziel der Arbeit ist ein Beweis der Substitutionsformel für Integrale von Differentialformen (Kroneckersche Integralformel) unter sehr allgemeinen, natürlichen Voraussetzungen und ohne die Hilfsmittel der kombinatorischen Topologie einschließlich simplizialer Approximationen. Zunächst wird die Substitutionsregel im gewöhnlichen Raum R^n in der Gestalt $\int_{f(X)} \alpha d\lambda_n = \int_X (\alpha\circ f)|\delta| d\lambda_n$, wobei λ_n das Lebesguesche Maß bedeutet, unter den folgenden Voraussetzungen abgeleitet: f sei eine umkehrbar eindeutige Abbildung der meßbaren Menge X von R^n in R^n, die in $X-N$ differenzierbar (im Stolzschen Sinne) ist, wobei $\lambda_n(N)=\lambda_n(f(N))=0$, es bezeichne δ die Funktionaldeterminante von f (die notwendig fast überall auf X eindeutig und meßbar ist) und α eine integrierbare reelle Funktion auf $f(X)$. Der Verf. entwickelt sodann sehr sorgfältig eine Meßbarkeits- und Integrationstheorie im Lebesgueschen Sinne für sogenannte orientierte C^λ-Mannigfaltigkeiten. Dies sind n-dimensionale lokaleuklidische Hausdorffsche Räume mit abzählbarer Basis nebst einem "Atlas", d.h. einem System von Koordinatentransformationen, die fast überall differenzierbar sind mit positiver Funktionaldeterminante. Insbesondere ist jede C^1-Mannigfaltigkeit und jede der vom Ref. [Math. Nachr. **11** (1954), 35–60; MR **15**, 692] betrachteten lokal dehnungsbeschränkten Mannigfaltigkeiten eine C^λ-Mannigfaltigkeit. Es werden Nullmengen, meßbare Mengen und meßbare Funktionen in einer C^λ-Mannigfaltigkeit definiert und zwar invariant gegenüber der Ersetzung eines Atlasses durch einen "äquivalenten". Da bei gegebenem abzählbarem Atlas $\mathscr{A}$ zwar der Tangentialraum fast überall existiert, die "Ausnahmemenge" aber von $\mathscr{A}$ abhängen kann, so bedarf es zur Definition des Begriffs einer Differentialform ω einer Klassenbildung: für jedes $\mathscr{A}$ hat ω einen durch $\mathscr{A}$ bestimmten Repräsentanten, der eine Differentialform im üblichen Sinne ist. Eine stetige Abbildung einer orientierten C^λ-Mannigfaltigkeit M in eine andere, M', gleicher Dimension, heißt eine λ-Abbildung, wenn sie hinsichtlich geeigneter abzählbarer Atlanten differenzierbar ist außer in einer Menge, die vermöge f in eine Nullmenge übergeht. Ist M' zusammenhängend und die λ-Abbildung f eigentlich, d.h. das Urbild jeder kompakten Menge kompakt, so läßt sich mittels der Methoden von Nagumo [Osaka Math. J. **2** (1950), 105–118; Amer. J. Math. **73** (1951), 485–495; MR **13**, 373, 150] der Abbildungsgrad d von f definieren und es gilt die Kroneckersche Integralformel $\int_M f^*\omega' = d\int_{f(M)} \omega'$, wenn die Differentialform ω' auf M' meßbar ist und die aus ω' vermöge f durch Substitution entstandene Differentialform $f^*\omega'$ auf M integrierbar ist. Aus $f(M)\neq M'$ folgt $d=0$.

K. Krickeberg (Heidelberg)

4626:

Cecconi, Jaures. Sulla derivazione delle funzioni normali di intervallo. Boll. Un. Mat. Ital. (3) **12** (1957), 200–204.

L'A., fondandosi sui risultati relativi alla derivazione delle funzioni additive d'insieme astratto ottenuti da C. Pauc [C. R. Acad. Sci. Paris **236** (1953), 1937–1939; MR **15**, 205] e dal recensore [Rend. Sem. Mat. Univ. Padova **23** (1954), 366–397; MR **16**, 345], estende i teoremi di Banach [Fund. Math. **6** (1924), 170–188] e Saks [Bull. Acad. Polska Sci. Lett. **3** (1926), 103–108] alle funzioni semi-additive (sub-additive o super-additive) definite **su** insiemi astratti.

Tali risultati sono stati ottenuti dall'A. indipendentemente da M. Pagni [Rend. Sem. Mat. Padova **25** (1956),

279–302; MR 18, 118] che, poco anteriormente, aveva trattato analoghi problemi con procedimenti non dissimili.

G. Fichera (Roma)

4627:

Christian, R. R. On order-preserving integration. Trans. Amer. Math. Soc. **86** (1957), 463–488.

The present paper is closely related to McShane's "Order-preserving maps and integration processes" [Ann. of Math. Studies, vol. 31, Princeton Univ. Press, 1953; MR 15, 19]. It provides another approach to order-integration.

§ I. $\mathfrak{X}=(\mathfrak{X}, \leqq)$: partially ordered set. "Directed floor" Φ: class of subsets of $\mathfrak{X}$, all of which are either directed by $\leqq$ or by $\geqq$. "First floor with base B": directed floor consisting of a single element $\mathfrak{M}$ such that $\wedge\mathfrak{M}=B$ ($\vee\mathfrak{M}=B$). "Tower with base B": finite ordered collection $\omega=\{\Phi_1, \cdots, \Phi_k\}$ of floors, all of which are directed either by $\leqq$ or by $\geqq$, such that Φ_1 is a first floor with base B, and to each $M\in\mathfrak{M}\in\Phi_n$, $1\leqq n<k$, there exists $\mathfrak{N}_M\in\Phi_{n+1}$ with $\wedge\mathfrak{N}_M=M$ ($\vee\mathfrak{N}_M=M$). A: $\prec$-directed set of elements a. A generalized sequence $\{X_a\}$ of elements of $\mathfrak{X}$ is "k-convergent to X", written $X=k\text{-}\lim X_a$, if there exist k-towers $\omega^{\leqq}=\{\Phi_1, \cdots, \Phi_k\}$ and $\omega^{\geqq}=\{\Psi_1, \cdots, \Psi_k\}$ with base X such that, for each $M\in\bigcup\Phi_i$ and $N\in\bigcup\Psi_i$, $i=1, 2, \cdots, k$, there exists $a(M, N)\in A$ for which $(a\prec a(M, N))\rightarrow(N\leqq X_a\leqq M)$. 1-convergence is McShane's o-convergence. $\{X_a\}$ converges to X, written $X=\lim X_a$, if $X=k\text{-}\lim X_a$ for some k. Theorems: Tower topology is stronger than G. Birkhoff's relative topology, which itself is stronger than interval topology. If $\mathfrak{X}$ is a complete lattice, the following statements are equivalent: (i) $\{X_a\}$ 1-converges to X; (ii) $\{X_a\}$ converges to X; (iii) $\liminf X_a=\limsup X_a=X$.

Partially ordered vector-spaces are investigated from the point of view of tower topology. "$\mathfrak{X}$ is Dedekind complete": every $\leqq$ [$\geqq$]-directed subset of $\mathfrak{X}$, which is bounded below [above] has an infimum [supremum]. Example: $\mathfrak{X}$ denotes the class of all bounded Hermitian operators X in Hilbert space $\mathfrak{H}$ partially ordered by $(X_1\leqq X_2)\Leftrightarrow((X_1(x), x)\leqq(X_2(x), x)$ for all $x\in\mathfrak{H})$.

§ II. $\mathfrak{X}$: Dedekind complete partially ordered vector-space. S: fundamental set. f: generic real-valued (finite) function on S. Σ^*: Boolean algebra of subsets of S ("measurable sets") with S as unit. "Simple function": function f assuming only a finite number of distinct values $\alpha_1, \cdots, \alpha_n$ and such that $f^{-1}(\alpha_i)\in\Sigma^*$ for $i=1, \cdots, n$. Σ: ideal of Σ^* (ideal of the "integrable sets"). χ_e: characteristic function of $e\in\Sigma$. μ: positive (meaning "non-negative") finitely additive function from Σ into $\mathfrak{X}$. For any subset D of S included in some set of Σ, $\mu^*(D)$ is defined as $\wedge\{\mu(e); e\supseteq D, e\in\Sigma\}$. "The generalized sequence $\{f_a\}$ converges to f in measure": for each $\varepsilon>0$ the set $E(a, \varepsilon)=\{s; |f_a(s)-f(s)|\geqq\varepsilon\}$ is eventually included in a set $e\in\Sigma$, and $\lim\mu^*(E(a, \varepsilon)=0$. "$f$ is μ-measurable": for each $e\in\Sigma$ there exists a generalized sequence of simple functions converging to $\chi_e f$ in measure. An "integrable simple function" is a function $f=\sum\alpha_i\chi_{e_i}$, $i=1, \cdots, k$, where the α_i are real numbers, $e_i\in\Sigma$ and $(i'\neq i'')\rightarrow(e_{i'}$ and $e_{i''}$ are disjoint); the "μ-integral" $\int_E f(s)d\mu(s)$ of a positive μ-measurable function f over $E\in\Sigma^*$ is defined as $\vee\int_E g(s)d\mu(s)$, where g denotes any integrable simple function such that $0\leqq g\leqq f$; the "μ-integral of an arbitrary μ-measurable f" is $\int_E f^+(s)d\mu(s)-\int_E f^-(s)d\mu(s)$, provided both integrals exist. The μ-integral enjoys the usual properties of an integral of Riemannian type. Conditions are given for a sequence $\{f_n\}$ of integrable (over S) functions converging in measure on S to an integrable function, to converge in the mean to the same function. Assuming Σ^* to be σ-closed and μ countably additive, then: (1) Every Σ^*-measurable function is μ-measurable; (2) sequences of Σ^*-measurable functions admit a dominated mean 1-convergence theorem; (3) the μ-integral is a (with respect to 1-convergence) countably additive set function.

§ III. $\mathfrak{X}$ as in § II. S: normal topological space. Σ: Boolean algebra generated by the closed subsets of S. $r(\Sigma)$: set of the positive finitely additive functions from Σ into $\mathfrak{X}$ such that $\mu(e)=\vee\mu(F)$, where $F\subseteq e$ is closed. $C(S)$: space of all real-valued bounded continuous functions on S. Theorems: (1) If T is a positive linear transformation from $C(S)$ into $\mathfrak{X}$, then there exists $\mu_T\in r(\Sigma)$ such that $T(f)=\int_S f(s)\mu_T(ds)$ for all $f\in C(S)$. Conversely, if $\mu\in r(\Sigma)$, then $T_\mu(f)=\int_S f(s)d\mu(s)$ defines a positive linear transformation from $C(S)$ into $\mathfrak{X}$, and the correspondence $T\Leftrightarrow\mu$ is reciprocal. (2) If $\mathfrak{X}$ is an algebra and T is multiplicative (i.e., $T(f_1f_2)=T(f_1)T(f_2)$ for all $f_1, f_2\in C(S)$), then $\mu(e_1\cap e_2)=\mu(e_1)\mu(e_2)$ for all $e_1, e_2\in\Sigma$. For the proof of Theorem (1) the author refers to A. D. Alexandroff, Mat. Sb. N.S. **9(51)** (1941), 563–628 [MR 3, 207], p. 577. {Reviewer's remark: Apparently only the Boolean ring Σ of subsets of S is needed, Σ^* being then defined as the class of subsets E of S such that $E\cap e\in\Sigma$ for all $e\in\Sigma$}.

Chr. Pauc (Nantes)

4628:

Eggleston, H. G. Tangential properties of Fréchet surfaces. Proc. Cambridge Philos. Soc. **54** (1958), 187–196.

Let (Ψ, H), $\Psi=(\Psi_1, \Psi_2, \Psi_3)$, be any continuous mapping from a closed disc H of the (x_1, x_2)-plane into E_3, let Π be the corresponding Fréchet surface and $L(\Pi)$ be its Lebesgue area; let $\Psi=lm$, $m: H\rightarrow\mathfrak{M}$, $l: \mathfrak{M}\rightarrow E_3$, be a monotone-light factorization of Ψ, and $\mathfrak{M}$ the associated hyperspace. If $\rho(p, q)$ is the usual Whyburn distance of two points $p, q\in\mathfrak{M}$, then the set $S(p, r)$ of all $q\in\mathfrak{M}$ with $\rho(p, q)<r$ is a sphere of center p and radius r on $\mathfrak{M}$. If $Q\subset\mathfrak{M}$ is any set, and A_δ, $\delta>0$, denotes any covering of Q of the type $[S(p_i, r_i), r_i<\delta, i=1, 2, \cdots]$, then a two-dimensional Hausdorff measure $\Xi(Q)$ can be defined by $\Xi(Q)=\lim\inf\sum\pi r_i^2$, where $\sum$ ranges over all $i=1, 2, \cdots$, Inf is taken over all A_δ, lim is taken as $\delta\rightarrow 0+$. A concept of approximate normal n to Π (and of approximate tangent plane) is then associated to all points p of a convenient set $T\subset\mathfrak{M}$. The author proves the following theorem. (I) For every (Ψ, H) we have $L(\Pi)=\Xi(T)$. The proof is based on a number of density theorems. {It appears that the following theorem is used, which is false and erroneously attributed to the reviewer. (R) For every (Ψ, H) with $L(\Pi)<+\infty$, there is another (Φ, H), $\Phi=(\Phi_1, \Phi_2, \Phi_3)$, which is Fréchet equivalent to Ψ, for which the vectors $V_s=(\partial\Phi_i/\partial x_s, i=1, 2, 3)$, $s=1, 2$, exist a.e. in H and $V_1\cdot V_2=0$, $|V_1|=|V_2|$ a.e. in H; and for which $L(\Pi)=(H)\int|V_1|^2dx_1dx_2$, and (*) Φ_1, Φ_2, Φ_3 are of bounded variation and absolutely continuous in the sense of Tonelli. For clarification of the author's misstatement see L. Cesari, "Surface Area" [Princeton Univ. Press, 1956; MR 17, 596], pp. 478, 487: the requirement (*), not contained in the reviewer's representation theorem [ibid., p. 544], makes statement (R) false. In the reviewer's opinion the error can be set right without jeopardizing the final result of the paper, by removal of requirement (*).}

L. Cesari (Baltimore, Md.)

FUNCTIONS OF A COMPLEX VARIABLE

See also 4543, 4742, 4787.

4629:

Springer, T. A. The Cauchy theorem. Simon Stevin **32** (1958), 68–79. (Dutch)

Let f be holomorphic in an open connected set G in the complex plane. The author defines $\int_k f(z)dz$ for every closed curve (continuous image of the unit circle) in G. He then proves Cauchy's theorem in the following form. $\int_k f(z)dz=0$ for every contractible curve k. Here a curve is called contractible if it is a "curvilinear triangle", that is, the restriction of a continuous map of a closed triangular region to the boundary. The proof is related to M. H. A. Newman's [Elements of the topology of plane sets of points, University Press, Cambridge, 1939; p. 154; for a review of the 1951 ed., see MR **13**, 483], but uses less topology. The author essentially extends Goursat's proof for a triangle to a "curvilinear triangle".

J. Korevaar (Madison, Wis.)

4630:

Havin, V. P. The separation of the singularities of analytic functions. Dokl. Akad. Nauk SSSR **121** (1958), 239–242. (Russian)

This paper deals with the problem of writing an analytic function as the sum of functions each of which has only some of the singular points of the original function. (1) Let F_0, F_1, F_2 be closed sets; G_0, G_1, G_2, their complements; $F_0=F_1\cup F_2$. If u_0 is analytic in G_0 then there are functions u_1 and u_2 such that $u_0(z)=u_1(z)+u_2(z)$ for z in G_0, and u_i is analytic on G_i $(i=1, 2)$. (2) Let u_0 be analytic in $|z|<1$; let $\int_0^\theta |u_0(re^{i\phi})|^p d\phi<C<\infty$, $p>1$, for some fixed θ; let $\int_0^1 |u_0(r)|dr$ and $\int_0^1 |u_0(re^{i\theta})|dr$ converge. Then there is a function Ψ in $L^p(0, \theta)$ such that

$$u_0(z)=\int_0^\theta \frac{\Psi(\phi)d\phi}{e^{i\phi}-z}+u_1(z),\ |z|<1,$$

where u_1 is analytic everywhere except for $|z|=1$, $0\leq \arg z\leq 2\pi$.

R. P. Boas, Jr. (Evanston, Ill.)

4631:

Cook, Erben, Jr. Divided differences in complex function theory. Amer. Math. Monthly **65** (1958), 17–24.

The function $F(z, a)=[F(z)-F(a)]/(z-a)$, $F(a, a)=F'(a)$ is the first divided difference of the analytic function $F(z)$; the nth divided difference $F(a_1, \cdots, a_{n-1}, z)$ is defined recursively. These are used to deduce elementary results, such as Taylor's formula with remainder, and additional theorems, of which the following is an example: If $f(z)$ is analytic in a region Ω, $\{a_k\}$ $(k=1, 2, \cdots)$ is any sequence in Ω, $\{z:|z-a|\leq R\}$ is a subset of Ω, $|a-a_k|\neq R$ for any k, and $M=\max_{|z-a|=R}|f(z)|$, then, for $|z-a|<R$ and all n,

$$|f(a_1, \cdots, a_n, z)|\leq \frac{MR}{(R-|z-a|)\prod_{k=1}^n |R-|a_k-a||}.$$

R. M. Redheffer (Los Angeles, Calif.)

4632:

Shibata, Kêichi. On boundary values of some pseudo-analytic functions. Proc. Japan Acad. **33** (1957), 628–632.

The author states and proves the following theorem. Let $\zeta=\varphi(z)$ be a quasiconformal mapping of the unit disk $|z|<1$ onto $|\zeta|<1$. A sufficient condition for φ to be absolutely continuous on the circumference $|z|=1$ is that $\varphi(z)$ be continuously differentiable and conformal with respect to some Riemannian metric $ds=|dz+h(z)d\bar{z}|$, where the function $h(z)$ is of norm $|h|\leq k<1$ and Hölder continuous: $|h(z_1)-h(z_2)|\leq \text{const}\ |z_1-z_2|^\alpha$, $0<\alpha\leq 1$. The proof makes use of a uniform approximation of the function $h(z)$ by functions $h_n(z)$ with the same properties but defined in the whole plane, and applies Ahlfors's mapping theory [Ann. Acad. Sci. Fenn. Ser. A. I. no. 206 (1955); MR **17**, 657; and lectures delivered at the Osaka University in February, 1956]. As an application, Fatou's theorem is extended to bounded pseudoanalytic functions in the unit disk with adequate continuity properties.

K. Strebel (Freiburg)

4633:

Parodi, Maurice. Sur la localisation des zéros des polynomes lacunaires. C. R. Acad. Sci. Paris **247** (1958), 391–393.

The author proves that all the zeros of the polynomial $f(z)=z^n+a_pz^{n-p}+\cdots+a_{n-1}z+a_n$, $p>1$, lie in the union of the interior of the unit circle and the interior of the curve $|z^p+a_p|=|a_{p+1}|+|a_{p+2}|+\cdots+|a_n|$. The proof depends upon the representation of $f(z)$ as a determinant and the use of Hadamard's sufficient condition for the non-vanishing of a determinant. Applications are made to the trinomial and quadrinomial equations.

M. Marden (Milwaukee, Wis.)

4634:

Schurrer, Augusta. On the location of the zeros of the derivative of rational functions of distance polynomials. Trans. Amer. Math. Soc. **89** (1958), 100–112.

A "distance polynomial" was defined by G. v. Sz. Nagy [Bull. Amer. Math. Soc. **55** (1949), 329–342; MR **10**, 702] as a polynomial of the m real variables $x_1, \cdots, x_m$ having the form

$$F(x_1, \cdots, x_m)=c\prod_{k=1}^n[(x_1-x_{1,k})^2+\cdots+(x_m-x_{m,k})^2],\ c>0,$$

and its "derivative" as

$$F'(x_1, \cdots, x_m)=(4F)^{-1}\sum_{k=1}^n(\partial F/\partial x_k)^2.$$

For such polynomials Nagy proved some theorems analogous to those due to Lucas, Jensen and Laguerre. In the present paper the author obtains the following as her principal result: Let F_j be a distance polynomial of degree n all of whose zeros lie in a given spherical region S_j. Then every finite zero of the derivative of the rational function $R=\prod_{j=0}^q F_j\prod_{j=q+1}^p (F_j)^{-1}$ lies in at least one of the regions specified by a set of $p+1$ inequalities. This result, a generalization of a theorem on the derivative of a rational function of a single complex variable, due to M. Marden [Trans. Amer. Math. Soc. **32** (1930), 81-109], is proved by a vector method which is a generalization of the method used by M. Marden [Bull. Amer. Math. Soc. **42** (1936), 400–405]. The author's result includes some of Nagy's as special cases and leads to spatial generalizations of various circle theorems on the derivative of a polynomial in a single complex variable due to Walsh.

M. Marden (Milwaukee, Wis.)

4635:

Bauer, Friedrich L.; and Frank, Evelyn. Note on formal properties of certain continued fractions. Proc. Amer. Math. Soc. **9** (1958), 340–347.

The Euclidean algorithm for given polynomials $P(x)$ and $p_{n-1}(x)$, of degrees n and $n-1$, respectively, and with leading coefficient unity, is formulated in terms of matrices. Thereafter, the equations of the algorithm are put into forms which lead to an S-fraction expansion and to

two J-fraction expansions which are essentially the even and odd parts of the S-fraction. A similar procedure for another division algorithm yields the Euler continued fraction expansion and a special case of the generalized Schur continued fraction expansion.

W. T. Scott (Evanston, Ill.)

4636:

Ilieff, Ljubomir. Ein Satz über analytische Nichtfortsetzbarkeit von Potenzreihen. C. R. Acad. Bulgare Sci. **10** (1957), 447–450. (Russian summary)

Let $f(z)=\sum_{n=0}^{\infty} c_n z^n$ with radius of convergence 1. The unit circle is a natural boundary if there are a sequence $\{n_\nu\}$ and a positive ε such that $\limsup |c_{n_\nu}|^{1/n_\nu}=1$; the numbers $c_{n_\nu-k}/c_{n_\nu}$ ($k=1, 2, \cdots, [\varepsilon n_\nu]$) do not have 1 as a limit point; and all the finite sequences $\{\gamma_k\}=\{c_{n_\nu-k}\}_{k=1}^{t_\nu}$, $t_\nu=[\varepsilon n_\nu]$, are "periodic" in the sense that, for some p, $\gamma_x=\gamma_{x+p}$ for $1\leqq x\leqq t_\nu-p$, with $[(t_\nu-1)/p]\geqq 2$ (so that there are at least two "periods" in each block of coefficients).

R. P. Boas, Jr. (Evanston, Ill.)

4637:

Havin, V. P. Analytic continuation of power series and Faber polynomials. Dokl. Akad. Nauk SSSR (N.S.) **118** (1958), 879–881. (Russian)

Let G be a simply connected region, containing ∞, whose boundary contains more than one point. Let $\Phi_k(z)=c_0^{(k)}+c_1^{(k)}z+\cdots+c_k^{(k)}z^k$ be the Faber polynomials for G, defined by $[\Phi(z)]^k=\Phi_k(z)+O(1/|z|)$, where $\Phi(z)=z+\alpha_0+\alpha_1 z^{-1}+\cdots$ maps G on $|w|>\rho$. Let $\phi(z)=\sum_{k=1}^{\infty} b_k z^{-k}$ converge in some neighborhood of ∞. The author shows that $\phi(z)$ is analytic in G if and only if

$$\limsup_{k\to\infty} |c_0^{(k)}b_1+\cdots+c_k^{(k)}b_{k+1}|^{1/k}\leqq\rho.$$

R. P. Boas, Jr. (Evanston, Ill.)

4638:

Heins, Maurice. The conformal mapping of simply-connected Riemann surfaces. II. Nagoya Math. J. **12** (1957), 139–143.

Using the method of subharmonic functions, the author is able to modify his earlier proof of the Riemann mapping theorem for simply-connected Riemann surfaces [Ann. of Math. (2) **50** (1949), 686–690; Nagoya Math. J. **9** (1955), 17–20; MR **11**, 93; **17**, 473]. His new version does not make use of Radó's theorem on the existence of a countable base for Riemann surfaces. A by-product is a new proof of this theorem.

K. Strebel (Freiburg)

4639:

Schoenberg, I. J. Some extremal problems for positive definite sequences and related extremal convex conformal maps of the circle. Nederl. Akad. Wetensch. Proc. Ser A **61**=Indag. Math. **20** (1958), 28–37.

Let $\varphi(t)$ be a positive mass distribution of total mass 1 on a circumference of radius 1, and let $\mu_\nu=\oint e^{i\nu t}d\varphi(t)$ ($\nu=0, \pm 1, \pm 2, \cdots$). Suppose that $\tilde\varphi$ has its entire mass contained in an arc $\alpha\leqq t\leqq\alpha+\lambda$ ($0<\lambda<2\pi$) and has moments $\tilde\mu_\nu$ such that $\tilde\mu_\nu=0$ if $\nu=n_1, n_2, \cdots$ ($1<n_1<n_2<\cdots$). Next let $\tilde F(z)=z+\sum_{\nu=1}^{\infty}\tilde c_{n_\nu}z^{n_\nu}$ map $|z|<1$ onto a schlicht and convex domain $\tilde D$ such that an arc $z=e^{it}$ ($\beta\leqq t\leqq\beta+\lambda$) maps onto a finite straight segment S in the boundary of $\tilde D$. The following two theorems are first proved: (1) For every distribution $\varphi(t)$ with the property $\mu_\nu=0$ for $\nu=n_1, n_2, \cdots$, the inequality $|\mu_1|\leqq|\tilde\mu_1|$ holds, clearly with equality if $\varphi\equiv\tilde\varphi$. (2) Let $F(z)=z+\sum_{\nu=1}^{\infty} c_{n_\nu}z^{n_\nu}$ map $|z|<1$ onto a schlicht, convex domain D. Then D contains the circle $|w|<|\tilde\mu_1|$. For $\tilde D_0$ the segment S is tangent to the circle $|w|=|\tilde\mu_1|$ at some point of S between its end points. By specializing $\tilde\varphi$ the author gives several applications of the above theorems. For example, he proves that the image of the unit circle under schlicht, convex mappings of the form $F(z)=z+c_2z^2+c_4z^4+c_6z^6+\cdots$ covers the circle $|w|<2/\pi$. Similarly, under schlicht, convex mappings of the form $F(z)=z+c_{k+1}z^{k+1}+c_{2k+1}z^{2k+1}+\cdots+c_{\nu k+1}z^{\nu k+1}+\cdots$, the image of the unit circle covers the circle $|w|<\Gamma^2((k+1)/k)/\Gamma((k+2)/k)$.

G. Springer (Lawrence, Kans.)

4640:

Read, Arthur H. A converse of Cauchy's theorem and applications to extremal problems. Acta Math. **100** (1958), 1–22.

Let W be a Riemann surface with closure $\overline{W}$ and analytic boundary curves C, let g be a bounded measurable function on C, and let ω range over the class of differentials analytic in $\overline{W}$. In a quite general formulation it is shown that if $\int_C g\omega=0$ for all ω, then g represents the boundary values of a function analytic in W. This theorem is used in conjunction with the Hahn-Banach theorem to establish the existence of solutions of conjugate extremal problems in the classes H_p and H_q for W, where $p^{-1}+q^{-1}=1$. Analyticity of the extremal functions on the boundary C is discussed. Reference is made to special cases of the extremal problems treated by the author which have already been treated in the literature.

P. R. Garabedian (Stanford, Calif.)

4641:

Bader, Roger; et Sörensen, Werner. Sur le problème de Cousin pour une surface de Riemann non compacte. Ann. Acad. Sci. Fenn. Ser. A. I, no. **250/1** (1958), 9 pp.

Let D be a finite Riemann surface with a smooth boundary D' and let U be a current whose support consists of a finite number of points. The author studies the equation

$$\Delta T=U,$$

where Δ is the Laplace-Beltrami operator. The above equation has a unique solution ω such that: ω is of class C^∞ outside of the support of U, $*d\omega$ and $\delta\omega$ vanish on D' and ω is orthogonal to the space of norm-finite harmonic forms.

J. J. Kohn (Waltham, Mass.)

4642:

Parreau, M. Théorème de Fatou et problème de Dirichlet pour les lignes de Green de certaines surfaces de Riemann. Ann. Acad. Sci. Fenn. Ser. A. I, no. **250/25** (1958), 8 pp.

In the paper, Riemann surfaces R with the following properties are considered: R possesses Green's functions g, every domain $g(p)>\lambda$, $\lambda>0$, is compact relative to R, and $\sum g(p_n)<\infty$, where p_n denote the points at which grad $g=0$. It is proved that if f is of bounded characteristic on R, i.e., if log $|f|$ admits a superharmonic majorant on R, then f tends to a limit on almost all Green's lines on R. This result is important in generalizing value distribution theory concerning functions of bounded characteristic in the unit disc to conformal mappings of a Riemann surface into another [Heins, Ann. of Math. (2) **61** (1955), 440–473; **62** (1955), 418–446; MR **16**, 1011; **17**, 726; Parreau, Ann. Fac. Sci. Univ. Toulouse (4) **19** (1955), 175–190; MR **18**, 291]. The author also proves the unique solvability on R of the Dirichlet problem for harmonic functions if the given boundary values are bounded and summable with respect to the harmonic measure.

O. Lehto (Helsinki)

4643:

Petersson, Hans. Explizite Konstruktion der automorphen Orthogonalfunktionen in den multiplikativen Differentialklassen. Math. Nachr. **16** (1957), 343–368.

Let Γ be a horocyclic group (Fuchsian group of the first kind) on the upper half-plane. The author wishes to study automorphic forms of dimension -2 on Γ. Essentially, his method is the following: he defines forms $F_r(z)$ on Γ of dimension $-r<-2$ with multipliers v_r. (This is easy, for the Poincaré series converges absolutely in this range of r.) Here v_r is a multiplier system for the dimension $-r$ so constructed that it tends continuously to a preassigned multiplier system v_2 of dimension -2. He then proves that, under certain conditions, $F_r(z)$ tends uniformly to a function $F_2(z)$, and that $F_2(z)$ is an automorphic form on Γ of dimension -2 with the assigned multipliers v_2.

In the first part of the paper, Petersson uses for F_r the Poincaré series of parabolic type: $G_{-r}(\tau, v, A, \Gamma, \nu)=\sum v^{-1}(M)(m, \tau+m_2)^{-r}\exp(2\pi i(\nu+\kappa)M\tau/N)$, where $M=(\dot m_1 \dot m_2)$ runs over an inequivalent system of matrices in $A\Gamma$. The theorem mentioned above was already proved by Petersson in 1948 [Math. Nachr. **1** (1948), 158–212, 218–257; MR **10**, 365, 525], but this time he proves it under weaker hypotheses.

He goes on to consider Poincaré series of elliptic type:

$$\Psi_{-r}(\tau, v, z, \Gamma, \nu)=\sum_{M\in\Gamma} v^{-1}(M)(m_1\tau+m_2)^{-r}(M\tau-\bar z)^{-r}\left(\frac{M\tau-z}{M\tau-\bar z}\right)^{\nu}.$$

Ψ_{-r} tends to Ψ_{-2}, $r\to 2$, where Ψ_{-2} is a function with the expected properties. In addition, Ψ_{-2} is discussed as a function of the parameter z.

J. Lehner (East Lansing, Mich.)

4644:

Kővári, Tamás. On a problem set by P. Turán. Mat. Lapok **7** (1956), 106–107. (Hungarian. Russian and English summaries)

The positive solution of the following problem of Turán is given: is there an integral function $f(z)$, for which the set of the roots of the functions $f(z), f'(z), \cdots, f^{(n)}(z), \cdots$ is everywhere dense on the whole complex plane?

Author's summary

4645:

Erdős, Pál. Remarks on a paper of T. Kővári. Mat. Lapok **7** (1956), 214–217. (Hungarian. Russian and English summaries)

Let $z_1, z_2, \cdots$ be an arbitrary sequence of complex numbers and $n_1<n_2<\cdots$ an arbitrary infinite sequence whose complementary sequence is infinite. The author proves that there exists an entire function $f(z)$ so that $f^{(n_k)}(z_k)=0$ for $k=1, 2, \cdots$.

The following two problems are raised: 1. Let $H_1, H_2, \cdots$ be an infinite sequence of sets. No H_k has a finite limit point. Does there always exist a sequence of integers $n_1<n_2<\cdots$ and an entire function $f(z)$ so that the roots of $f^{(n_k)}(z)=0$ contain H_k for $k=1, 2, \cdots$? 2. Does there exist an entire function $f(z)$ so that for every sequence $n_1<n_2<\cdots$ the union of the set of all the roots of $f^{(n_k)}(z)=0$ is everywhere dense in the plane?

From the author's summary

4646:

Singh, S. K. On exceptional values of entire function. J. Math. Soc. Japan **10** (1958), 217–220.

The entire function $f(z)$ of positive finite order is said to have α as a V-exceptional value if

$$\Delta(\alpha)=1-\liminf N(r, \alpha)/T(r)>0;$$

as an S-exceptional value if

$$\liminf\frac{\log M(r)}{n(r, \alpha)\psi(r)}>0,$$

with $\log x=o(\psi(x))$; as an E-exceptional value if α is S-exceptional with ψ increasing and $\int^{\infty}\{x\psi(x)\}^{-1}dx$ convergent; and as an N-exceptional value if

$$1-\limsup N(r, \alpha)/T(r)>0.$$

The author shows that an S-exceptional value is V-exceptional with $\Delta(\alpha)=1$; that if 0 is an E-exceptional value then $m(r)\to 0$ (where $m(r)$ is the minimum modulus); and that if f has an N-exceptional value then $m(r)$ is bounded.

R. P. Boas, Jr. (Evanston, Ill.)

4647:

Mikhail, M. N. The behaviour of the random meromorphic function at its poles. Nederl. Akad. Wetensch. Proc. Ser. A **60**=Indag. Math. **19** (1957), 590–597.

The author continues previous investigations concerning a suitably defined family of random meromorphic functions. In part II of his investigations [Indag. Math. **19** (1957), 96–103; MR **19**, 259], the author considered the behavior of a meromorphic function of this family in the neighborhood of its zeros; he now considers the behavior of this function in the neighborhood of its poles. His results are not totally unexpected, since an inspection of his definitions reveals no essential difference between a function $g(z, t)$ (of the family considered) and its reciprocal $1/g(z, t)$.

A. Edrei (Syracuse, N.Y.)

4648:

Constantinescu, Corneliu. Über eine Klasse meromorpher Funktionen, die höchstens einen defekten Wert besitzen können. Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.) **1(49)** (1957), 131–140.

The following theorem which is related to earlier work of Valiron [Rend. Circ. Mat. Palermo **49** (1925), 415–421] and Tumura [Proc. Imp. Acad. Tokyo **17** (1941), 65–69; MR **2**, 358] is established. Let M_h denote the class of meromorphic functions w in the finite plane which satisfy

$$\liminf_{r\to\infty}\frac{T(r, w)}{(\log r)^2\log\log r}\leqq h<+\infty.$$

Then the functions of class M_h possess at most two defective values for $h=\pi^{-2}$ and at most one defective value for $h=2/(3\pi^2)$. If $h=(3\pi^2)^{-1}$ and $w\in M_h$ has the defective value a, then there exists a sequence of circles $|z|=r_n$, $\lim r_n=\infty$ on which w tends uniformly to a. The functions $w\in M_h$, $h=(3\pi^2)^{-1}$, which possess a defective value have at most one asymptotic value. Use is made of the principle of harmonic measure and an extension of a lemma of Tumura [op. cit.].

M. H. Heins (Urbana, Ill.)

4649:

Montel, Paul. Sur les valeurs algébriques des fonctions analytiques. Ann. Acad. Sci. Fenn. Ser. A. I, no. **250/24** (1958), 8 pp.

Let C: $y=f(z)$ be a meromorphic curve and Γ an algebraic curve defined by the equation $A_0u^m+A_1u^{m-1}+\cdots+A_m=0$, where the coefficients A_i are polynomials in z, and $m\geqq 3$. The study of the points of intersection of C and Γ leads to problems which generalize those pertaining to Picard's theorem. The classical situation arises if Γ consists of three lines parallel to the z-axis.

In this paper, it is assumed that C and Γ have at most p points in common. The main result is expressed as follows.

Consider the family of functions meromorphic in a domain D and taking at most p times the values of an algebraic function with at least three branches. In every domain interior to D, the functions then take a bounded number of times the values of any algebraic function v, with the only exception that a limit function of the family coincides with a branch of v.

Under the additional restriction that the functions of the family are regular, two theorems, corresponding to those of Landau and Schottky in the classical case, are then established. *O. Lehto* (Helsinki)

4650:

Dundučenko, L. E. Über einige Klasse Funktionen, welche im Kreise $|z| \leqq 1/\sqrt{2}$ schlicht sind. Bul. Inst. Politehn. Iaşi (N.S.) **3** (1957), 37–38. (Russian. German and Romanian summaries)

Tchakaloff [C. R. Acad. Sci. Paris **242** (1956), 437–439; MR 17, 724] proved that if $\alpha(t)$ is nondecreasing in $[0, 2\pi]$, then the function

$$w=\frac{1}{2\pi}\int_0^{2\pi}\frac{z\,d\alpha(t)}{1-e^{-it}z}=\sum_{n=1}^{\infty}a_nz^n$$

is univalent in $|z|<1/\sqrt{2}$.

The author announces a number of theorems about functions defined by this type of integral, with $\alpha(2\pi)-\alpha(0)=2\pi$. These include bounds for $|w|$, $|w'|$, $|\arg w|$, $|\arg w'|$, and $|a_n|$. *A. W. Goodman* (Lexington, Ky.)

4651:

Sakaguchi, Kôichi. On functions starlike in one direction. J. Math. Soc. Japan **10** (1958), 260–271.

Let

$$(1)\qquad f(z)=\sum_1^{\infty}a_nz^n\ \ (a_1=1)$$

be regular for $|z|<1$. The author shows that if $f(z)$ is univalent and star-like with respect to the origin in $|z|<1$, then

$$(2)\qquad f(z)-f(-z)=2\sum_0^{\infty}a_{2n+1}z^{2n+1}$$

is also star-like for $|z|<(3-2\sqrt{2})^{\frac{1}{2}}$ and convex for $|z|<(11-2\sqrt{30})^{\frac{1}{2}}$, while the partial sums (3) $\sum_{n=0}^{m}a_{2n+1}z^{2n+1}$ are all star-like in $|z|<\frac{1}{2}$ and convex for $|z|<1/(3\sqrt{3})$.

If $f(z)$ in (1) is convex for $|z|<1$, then the function in (2) is star-like for $|z|<1$ and convex for $|z|<(3-2\sqrt{2})^{\frac{1}{2}}$. Moreover, the partial sums (3) in this case are star-like for $|z|<1/\sqrt{3}$ and convex for $|z|<1/3$. The bounds are all sharp.

If $f_k(z)=\sum_{n=0}^{\infty}a_{kn+1}z^{kn+1}$ $(a_1=1)$ is regular and star-like in the direction of one ray, then it is shown that $f_k(z)$ is convex for $|z|^k<c-(c^2-1)^{\frac{1}{2}}$, where $2c=3k+6+(9k^2+32k)^{\frac{1}{2}}$. The result is sharp.

If $f(z)$, given as in (1), is regular and star-like in the direction of the diametral line, then $f(z)-f(-z)$ is also star-like in the direction of the diametral line, and is star-like with respect to the origin for $|z|<(3-2\sqrt{2})^{\frac{1}{2}}$ and convex for $|z|<(11-2\sqrt{30})^{\frac{1}{2}}$.

The methods used are classical and involve essentially the construction of appropriate regular functions with positive imaginary part.

M. S. Robertson (New Brunswick, N.J.)

4652:

Šlionskiĭ, G. G. On the extremal problems for differentiable functionals in the theory of univalent functions. Vestnik Leningrad. Univ. **13** (1958), no. 13, 64–83. (Russian. English summary)

"The author considers extremal problems for various classes of univalent functions in general functional form. Using variational methods based on Golusin's theorem he investigates the properties of extremal functions and establishes the differential equations for these functions. Applying Loewner's method of parametric representation of univalent functions, the author generalizes these results and obtains also some additional information about Loewner's equations corresponding to the extremal functions." (From the author's summary)

A. W. Goodman (Lexington, Ky.)

4653:

Singh, Vikramaditya. Some extremal problems for a new class of univalent functions. J. Math. Mech. **7** (1958), 811–821.

Let S denote the class of functions f which are regular and univalent in $|z|<1$, real on the real axis, whose image domain contains the unit circle, and which are normalized by $f(0)=0$ and $f(1)=1$. The author proves that, for $f\in S$, and ζ real and in the unit circle, $|f(\zeta)|\leqq 4|\zeta|/(1+\zeta)^2$, equality being attained only for the Koebe function. Furthermore, for $f\in S$ and $f(z)=a_1z+a_2z^2+\cdots$, the following sharp inequality holds: $a_2/a_1\leqq 2(\sqrt{2}-1)^2$. The latter result is proved by variational methods. He also gives the extremal function for which a_2/a_1 attains its upper bound. *G. Springer* (Lawrence, Kans.)

4654:

Wintner, Aurel. The theorem of Eneström and the extremal functions of Landau-Schur. Math. Scand. **5** (1957), 236–240.

Let E be the class of functions $f(z)=\sum_0^{\infty}c_kz^k$, regular and such that $|f(z)|<1$ for $|z|<1$. Let $s_n(z)$ denote the nth partial sum of the above power series. E. Landau [Darstellung und Begründung einiger neuerer Ergebnisse der Funktionentheorie, 2nd ed., Springer, Berlin, 1929] determined the extremal functions $f_n^*(z)=\sum_m c_m^*(n)z^m\in E$ for which the extremum value $\sup_{f\in E}\sup_{|z|<1}s_n(z)$ is obtained. The author shows, from a more general point of view, that $c_m^*(n)>0$, $0\leqq m\leqq n$, $c_{n+1}^*(n)<0$, $n>0$. {A reference to equation (14), made on page 239, is misprinted as (4).} *E. Reich* (Minneapolis, Minn.)

FUNCTIONS OF SEVERAL COMPLEX VARIABLES, COMPLEX MANIFOLDS

4655:

Gröbner, Wolfgang. L'inversione di un sistema di funzioni analitiche mediante serie di Lie. Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. (8) **24** (1958), 386–390.

The system $X_i=\varphi_i(Y_1,\cdots,Y_n)$ $(i=1,\cdots,n)$, X_i, Y_i complex variables, φ_i holomorphic, can be inverted by means of a series containing the linear operators $D_k=\sum\psi_{jk}(y)\partial/\partial y_j$, where $(\psi_{ij}(y))$ is the matrix inverse to the Jacobian at $Y_i=y_i$. The D_k commute: $D_kD_l=D_lD_k$. The method is used to calculate the solutions of systems of equations and the characteristics of first-order, ordinary differential equations. In the particular case $\varphi(Y_1)=0$, the series obtained begins with the two terms in Newton's formula.

C. D. Calsoyas (Livermore, Calif.)

4656:

✱Atiyah, M. F. Complex analytic connections in fibre bundles. Symposium internacional de topología algebraica [International symposium on algebraic topology], pp. 77–82. Universidad Nacional Autónoma de México and UNESCO, Mexico City, 1958. xiv+334 pp.

This is a summary of results in Trans. Amer. Math. Soc. **85** (1957), 181–207 [MR **19**, 172].

4657:

Stein, Karl. Die Existenz komplexer Basen zu holomorphen Abbildungen. Math. Ann. **136** (1958), 1–8.

Der Verf. bezeichnet mit X, Y, Y_1 usw. (normale) komplexe Räume und untersucht holomorphe Abbildungen f: $X\to Y$. Er nennt eine holomorphe Abbildung f_1: $X\to Y_1$ von f abhängig, wenn für die Abbildung $\alpha: x\to(f(x), f_1(x))$: $X\to Y\times Y_1$ gilt: $\dim_x f^{-1}f(x)=\dim_x\alpha^{-1}\alpha(x)$ für $x\in X$. Dieser Begriff stimmt auf komplexen Mannigfaltigkeiten mit der sonst üblichen Definition überein. Eine komplexe Basis zu der Abbildung f ist sodann ein Paar (f^*, X^*), für das folgendes gilt: 1) X^* ist ein komplexer Raum, 2) f^* ist eine surjective holomorphe Abbildung $X\to X^*$, die von f abhängig ist, 3) es gibt zu jeder von f abhängigen holomorphen Abbildung f_2: $X\to Y_2$ eine holomorphe Abbildung f_2^*: $X^*\to Y_2$, so dass $f_2=f_2^*f^*$ ist. Natürlich wird es nicht immer zu f eine komplexe Basis geben. Existiert sie jedoch, so ist sie bis auf Äquivalenz eindeutig bestimmt. — Für viele Fragen in der komplexen Analysis ist die Existenz komplexer Basen von wesentlicher Bedeutung. Der Verf. hat deshalb in früheren Arbeiten mehrere Kriterien angegeben. In der vorliegenden Arbeit wird gezeigt: Ist $r=\operatorname{codim}_x f^{-1}f(x)$ unabhängig von $x\in X$, so existiert (f^*, X^*). Es gilt: X^* ist reindimensional und $\dim X=r$.

H. Grauert (Princeton, N.J.)

4658:

✱Cartan, Henri. Espaces fibrés analytiques. Symposium internacional de topología algebraica [International symposium on algebraic topology], pp. 97–121. Universidad Nacional Autónoma de México and UNESCO, Mexico City, 1958. xiv+334 pp.

This paper consists mainly of an exposition of some results of H. Grauert [cf. the reviews below].

The author defines the notion of an analytic space in the language of local categories. The category of models $\mathscr{M}$ consists of analytic subsets of complex euclidean space and of holomorphic maps between them. $\tilde{\mathscr{M}}$, the local category of analytic spaces, is the local category derived from the functor T: $\mathscr{M}\to\mathscr{T}$, where $\mathscr{T}$ is the category of Hausdorff spaces [cf. #4837]. Thus, each object of $\tilde{\mathscr{M}}$ is locally isomorphic to an object of $\mathscr{M}$.

Let E be an analytic fibre space whose base space is an analytic space X and whose fibres are isomorphic to a complex Lie group G. Let $\mathscr{E}^a$ and $\mathscr{E}^c$ be the sheaves of germs of analytic and continuous sections of E, respectively. Then the inclusion map $\mathscr{E}^a\to\mathscr{E}^c$ induces the map (1) $H^1(X, \mathscr{E}^a)\to H^1(X, \mathscr{E}^c)$. Grauert's fundamental result is that (1) is an epimorphism if X is holomorphically complete.

There is a natural one to one correspondence between the elements of $H^1(X, \mathscr{E}^c)$ and a class of (topological) fibre spaces; these are called E-principal fibre spaces (in the case $E=X\times G$ the E-principal fibre spaces are the principal fibre spaces over X with group G). Similarly, $H^1(X, \mathscr{E}^a)$ corresponds to analytic E-principal fibre spaces. Thus the above result implies that there is a natural correspondence between analytic E-principal fibre spaces and topological E-principal fibre spaces over a holomorphically complete space X. Furthermore, roughly speaking, the result implies that the standard topological constructions on these fibre spaces can be realized analytically.

J. J. Kohn (Waltham, Mass.)

4659:

Grauert, Hans. Approximationssätze für holomorphe Funktionen mit Werten in komplexen Räumen. Math. Ann. **133** (1957), 139–159.

The question of approximating a function holomorphic in a domain $\mathfrak{R}$ by polynomials or by functions holomorphic in a larger domain $\check{\mathfrak{R}}$, which for one complex variable is answered by the theorem of Runge, is for several variables a deep and central one. It was first solved by H. Behnke, K. Stein and A. Weil: Let $\mathfrak{R}$ and $\check{\mathfrak{R}}$, $\mathfrak{R}\subset\check{\mathfrak{R}}$, be locally schlicht domains of holomorphy; then a necessary and sufficient condition for the approximability is the analytic-topological condition that "$\mathfrak{R}$ can be holomorphically expanded to $\check{\mathfrak{R}}$".

The author considers the following wide generalization: Let $\mathfrak{R}$ and $\check{\mathfrak{R}}$ be "holomorphically complete complex spaces" (that includes, in particular, Stein manifolds). Let $\mathfrak{R}$ be holomorphically expandable to $\check{\mathfrak{R}}$. Let $\mathfrak{W}$ be an arbitrary complex space. Consider the holomorphic mappings $F:\mathfrak{R}\to\mathfrak{W}$ and $\check{F}:\check{\mathfrak{R}}\to\mathfrak{W}$. Then under what (additional) conditions can every mapping F be approximated by mappings $\check{F}$ (uniformly in every compact subset of $\mathfrak{R}$)? The case where $\mathfrak{W}$ is a Riemann surface is treated in detail. In particular, it is shown that if $\mathfrak{W}$ is a Riemann surface of hyperbolic type there are already additional conditions; that is, in contrast to the case of complex-valued functions ($\mathfrak{W}$ the complex plane), the holomorphic expandibility of $\mathfrak{R}$ to $\check{\mathfrak{R}}$ alone is not enough.

The main result is: Let $\mathfrak{W}=L$ be a complex Lie group. Let $F(r):\mathfrak{R}\to L$ be a holomorphic mapping. If there exists a continuous family $F(r, t)$, $0\leq t\leq 1$, of holomorphic mappings $\mathfrak{R}\to L$ such that $F(r, 0)\equiv 1\in L$, $F(r, 1)=F(r)$, then $F(r)$ can be approximated (uniformly in every compact subset of $\mathfrak{R}$) by holomorphic mappings $\check{\mathfrak{R}}\to L$.

The results of this paper have deep and important consequences in the theory of analytic fibre bundles that the author has obtained in two further papers [reviewed below]. For these applications, the author shows: The above result is true even if $F(r)$ is dependent upon an additional parameter $\mathfrak{t}\in\mathfrak{T}$ ($\mathfrak{T}$ a compact space), such that $F(r)$ is only continuous for certain values of $\mathfrak{t}$, and $F(r)\equiv 1\in L$ for others. And instead of $F(r)$ and $\check{F}(r)$ having values in L, it is permissible that $F(r)$ and $\check{F}(r)$ be sections in a complex analytic fibre bundle over $\mathfrak{R}$ and $\check{\mathfrak{R}}$, respectively, with L as fibre and the group of automorphisms of L as structure group. (A mapping F: $\mathfrak{R}\to L$ can be considered as a section in the trivial bundle $\mathfrak{R}\times L$.)

H. J. Bremermann (Berkeley, Calif.)

4660:

Grauert, Hans. Holomorphe Funktionen mit Werten in komplexen Lieschen Gruppen. Math. Ann. **133** (1957), 450–472.

This paper, which is part of the author's "Habilitationsschrift", contains profound results about holomorphic functions on complex spaces with values in a complex Lie group.

Let $\mathfrak{R}$ be a complex space, L and L^* complex Lie

groups such that L^* is a group of automorphisms of L. Let $L(\mathfrak{R}, L^*)$ be a complex-analytic fibre bundle over $\mathfrak{R}$ with L as fibre and L^* as structure group. Let $\mathfrak{S}^a$ and $\mathfrak{S}^c$ be the sheaves of germs of holomorphic and continuous sections in $L(\mathfrak{R}, L^*)$ respectively. Let i^* be the homomorphism of the cohomology sets $H^1(\mathfrak{R}, \mathfrak{S}^a)$ into $H^1(\mathfrak{R}, \mathfrak{S}^c)$. Then the following fundamental theorem is shown: (1) If $\mathfrak{R}$ is holomorphically complete (that includes Stein manifolds), then i^* is an isomorphism onto. This result solves a problem posed by H. Cartan: Two analytic fibre bundles over a complex space are equivalent analytically if they are equivalent topologically. To every topological complex fibre bundle there exists an analytical fibre bundle equivalent to it. [Cf. #4658 above.]

The proof of (1) uses the so-called "Laurent separation" for holomorphic sections in fibre bundles $L(\mathfrak{R}, L^*)$ and approximation and deformation theorems. In particular it is shown: (2) Let $\mathfrak{R}$ and $\check{\mathfrak{R}}$ be holomorphically complete spaces. Let $\mathfrak{R}$ be a sub-region of $\check{\mathfrak{R}}$ that is holomorphically convex with respect to $\check{\mathfrak{R}}$. If then $F(r)$ is a holomorphic section over $\mathfrak{R}$ in $L(\mathfrak{R}, L^*)$ that can be approximated by sections continuous over $\check{\mathfrak{R}}$, then $F(r)$ can be approximated by sections holomorphic over $\check{\mathfrak{R}}$.

The proof of (2) is based on the following result: Let $H^0(\mathfrak{R}, \mathfrak{S}^a)$ and $H^0(\mathfrak{R}, \mathfrak{S}^c)$ be the groups of holomorphic and continuous sections in $L(\mathfrak{R}, L^*)$ respectively (which are the same as the groups of sections over $\mathfrak{R}$ in $\mathfrak{S}^a$ and $\mathfrak{S}^c$). Let $\tilde{H}^0(\mathfrak{R}, \mathfrak{S}^a)$ and $\tilde{H}^0(\mathfrak{R}, \mathfrak{S}^c)$ be the groups of holomorphic and continuous deformation classes of $H^0(\mathfrak{R}, \mathfrak{S}^a)$ and $H^0(\mathfrak{R}, \mathfrak{S}^c)$. Then the mapping $\tilde{H}^0(\mathfrak{R}, \mathfrak{S}^a) \to \tilde{H}^0(\mathfrak{R}, \mathfrak{S}^c)$ generated by the injection $\mathfrak{S}^a \to \mathfrak{S}^c$ is a homomorphism. [Cf. #4662 below.]

From (2) the author derives, by means of a rather technical lemma: If $\mathfrak{R}$ is a holomorphically complete space, then $\tilde{H}^0(\mathfrak{R}, \mathfrak{S}^a) \to \tilde{H}^0(\mathfrak{R}, \mathfrak{S}^c)$ is an isomorphism. The lemma is proved by a method of the author by which local deformations can be joined together to global deformations. In order to apply this method the results are proved from the beginning for sections $F(r, t)$ (so-called (ε, h)-functions) in $L(\mathfrak{R}, L^*)$ that depend upon an additional parameter $t \in \mathfrak{T}$ ($\mathfrak{T}$ a compact space).

H. J. Bremermann (Berkeley, Calif.)

4661:

Grauert, Hans. Analytische Faserungen über holomorph-vollständigen Räumen. Math. Ann. **135** (1958), 263–273.

Ce travail est le troisième et dernier d'une série; les démonstrations difficiles ont déjà été données dans les deux premiers [analysés ci-dessus]. On rassemble ici quelques résultats.

Il s'agit de fibrés analytiques dont la base B et la fibre F sont des espaces analytiques complexes, et dont le groupe structural est un groupe de Lie complexe L. Leur classification équivaut évidemment à celle des fibrés analytiques principaux de base B et de groupe L. Comme connu, l'ensemble des classes de tels fibrés est en correspondance bijective avec l'ensemble de cohomologie $H^1(B, \mathscr{L}^a)$, où $\mathscr{L}^a$ désigne le faisceau des germes d'applications holomorphes de B dans L; les classes de fibrés topologiques correspondent aux éléments de $H^1(B, \mathscr{L}^c)$, où $\mathscr{L}^c$ désigne le faisceau des germes d'applications continues de B dans L. L'inclusion $\mathscr{L}^a \to \mathscr{L}^c$ définit une application $H^1(B, \mathscr{L}^a) \to H^1(B, \mathscr{L}^c)$. Le théorème essentiel de Grauert (Satz 2) affirme que si B est un espace holomorphiquement complet, cette application est bijective. L'Auteur annonce une généralisation de ce résultat, dans le cadre d'une théorie (non publiée) des faisceaux analytiques cohérents non nécessairement abéliens.

Voici quelques applications. Si B est holom. complet, et si L_1 est un sous-groupe de Lie complexe de L, la restriction du groupe structural de L à L_1 est possible analytiquement dès qu'elle l'est topologiquement, par ex. si L/L_1 est contractile. Lorsque L_1 est contractile et B holom. complet, la trivialité d'un fibré analytique principal P de base B et de groupe L équivaut à l'existence d'une section continue de P/L_1, fibré de base B et de fibre L/L_1. On en déduit: si un fibré analytique, dont la base B est holom. complète, a pour fibre F une variété analytique complexe, et si son groupe structural laisse fixe un point $x_0 \in F$, X est trivial si et seulement si le fibré associé ayant pour fibre l'espace tangent à F en x_0 est trivial.

Autres résultats: tout fibré analytique dont la base est holom. complète et contractile est trivial. Tout fibré analytique dont le groupe structural est connexe et dont la base B est une variété analytique complexe de dimension 1, connexe et non compacte, est trivial. Toute variété de Stein de dimension $\geqq 1$ possède un champ holomorphe de vecteurs tangents partout $\neq 0$. Pour qu'une variété de Stein soit analytiquement parallélisable, il faut et il suffit qu'elle soit topologiquement parallélisable. Etc.

H. Cartan (Paris)

4662:

Frenkel, Jean. Cohomologie non abélienne et espaces fibrés. Bull. Soc. Math. France **85** (1957), 135–220.

Im Jahre 1950 zeigte H. Cartan, daß sich die Okaschen Sätze über die Cousinschen Probleme in der Sprache der Faserbündel formulieren lassen. Über Steinschen Mannigfaltigkeiten wird die analytische Äquivalenz von Faserbündeln, deren Strukturgruppe die additive oder die multiplikative Gruppe der komplexen Zahlen ist, von der topologischen Äquivalenz impliziert. J. P. Serre hat dieses Resultat auf den Fall beliebiger zusammenhängender abelscher Strukturgruppen verallgemeinert. Der Verf. der vorl. Arbeit untersucht nun analytische Faserbündel, bei denen die Strukturgruppe eine auflösbare oder nilpotente komplexe Liesche Gruppe ist. Er gewinnt folgende Hauptresultate (X immer eine parakompakte komplexe Mannigfaltigkeit, V ein komplex-analytisches Geradenbündel über X, $\mathfrak{V}$ die Garbe der Keime von holomorphen Schnittflächen in V, $\mathfrak{C}$ die Garbe der Keime holomorpher Funktionen): 1) Es sei $H^1(X, \mathfrak{V})=0$ für alle V über X. Dann sind zwei topologisch äquivalente analytische Faserbündel, deren (gleiche) Strukturgruppe zusammenhängend und auflösbar ist, auch analytisch äquivalent. 2) Es sei $H^2(X, \mathfrak{V})=0$ für alle V über X. Dann ist jedes topologische Faserbündel über X, dessen Strukturgruppe zusammenhängend und auflösbar ist, zu einem komplex-analytischen Faserbündel äquivalent. 3) Es sei $H^1(X, \mathfrak{C})=0$. Dann ist über X jedes topologisch triviale Faserbündel mit auflösbarer Strukturgruppe auch analytisch trivial. Ist die Strukturgruppe nilpotent, so gilt sogar die Aussage 1). 4) Es sei $H^2(X, \mathfrak{C})=0$. Dann ist über X jedes topologische Faserbündel mit nilpotenter Strukturgruppe zu einem analytischen Faserbündel äquivalent. — In den Aussagen 1) und 2) braucht nicht gefordert zu werden, daß G zusammenhängend ist, wenn die Fundamentalgruppe von X verschwindet. Analoge Sätze werden für reell-analytische Mannigfaltigkeiten und reell-analytische Faserbündel angegeben. — Der Beweis der Hauptaussagen stützt sich im wesentlichen auf eine Untersuchung von komplex-analytischen Faserbündeln L, deren Faser eine Liesche Gruppe L_0 und deren

Strukturgruppe eine Gruppe von Automorphismen von L_0 ist. Da L ein Bündel von Gruppen ist, sind die Garben $\mathfrak{L}_a$, $\mathfrak{L}_c$ der Keime von holomorphen bzw. stetigen Schnittflächen in L Garben von (nicht notwendig abelschen) Gruppen. Die Sätze 1)–4) sind Aussagen über den durch die Injektion $i: L_a \to \mathfrak{L}_c$ erzeugten Homomorphismus $i^*: H^1(X, \mathfrak{L}_a) \to H^1(X, \mathfrak{L}_c)$. Mit $\bar{H}^0(X, \mathfrak{L}_a)$ bzw. $\bar{H}^0(X, \mathfrak{L}_c)$ bezeichnet der Verf. Mengen von Klassen analytischer [stetiger] Schnittflächen in L. Zwei Schnittflächen gehören der gleichen Klasse an, wenn sie über analytische [stetige] Schnittflächen aufeinander deformiert werden können. Der Verf. leitet über die Abbildung $i^*: \bar{H}^0(X, \mathfrak{L}_a) \to \bar{H}^0(X, \mathfrak{L}_c)$ an sich interessante Sätze her, die wesentlich zum Beweis der Hauptresultate herangezogen werden (vgl. Théorème III.1 und Lemme 24.1, auch Théorème III.2). — Es sei noch eine kurze Inhaltsübersicht gegeben. Im I. Kapitel der Arbeit wird die Kohomologietheorie für Garben von nicht-abelschen Gruppen entwickelt. Es werden exakten Garbensequenzen $0 \to F' \to F \to F'' \to 0$ Kohomologiesequenzen zugeordnet. Dabei werden die Fälle, daß F' Normalteilergarbe, abelscher Normalteiler usw. von F ist, gesondert behandelt. Im II. Kapitel schließt sich eine Untersuchung über topologische Faserbündel an. Auch hier werden sehr allgemeine und abstrakte Resultate gewonnen. Das III. und letzte Kapitel bringt sodann die analytischen Faserbündel und die vorne angegebenen Hauptresultate. Es wird zunächst wieder eine allgemeine Untersuchung durchgeführt. Der Verf. erhält folgende interessante Aussage: Es sei X eine kompakte komplexe Mannigfaltigkeit. Dann sind zwei komplex analytische Faserbündel über X mit $O(n, C)$ (oder $Sp(n, C)$) als Strukturgruppe, die bzgl. der allgemeinen linearen Gruppe analytisch äquivalent sind, analytisch äquivalent. — Die letzten Seiten der Arbeit bringen Beispiele von komplexen Mannigfaltigkeiten X, die den Voraussetzungen der Sätze 1)–4) genügen. Zunächst sind das die Steinschen Mannigfaltigkeiten. Durch Anwendung von an sich interessanten Resultaten über Fréchetsche Garben gelingt es jedoch eine grössere Klasse anzugeben. Der Verf. errechnet die Kohomologiegruppen $H^\nu(X, \mathfrak{B})$ in dem Falle, daß X ein Produkt $X = U \times \prod_\iota X_\iota \times \prod_\lambda Y_\lambda$ ist, in dem U eine Steinsche Mannigfaltigkeit, X_ι gewisse Gebiete eines komplexen Zahlenraumes und Y_λ komplex projektive Räume bezeichnen. Er beweist damit aufs Neue (sehr einfach) die Sätze von Serre und Kodaira über die Kohomologiegruppen $H^\nu(P, \mathfrak{B})$ der komplex projektiven Räume P. Da $H^\nu(X, \mathfrak{B})$ für die Produktgebiete in vielen Fällen verschwindet, erhält er eine Reihe von Beispielen, in denen die Aussagen 1)–4) richtig sind. — Es bleibt zu erwähnen, daß die gemachten Voraussetzungen wesentlich sind. Ist die Strukturgruppe nicht auflösbar, so sind die Sätze 1)–4) i.a. falsch. Dagegen sind sie für allgemeine komplex-analytische Faserbündel richtig, wenn man voraussetzt, daß der Basisraum X eine Steinsche Mannigfaltigkeit oder ein holomorph vollständiger Raum ist [vgl. #4661].

H. Grauert (Princeton, N.J.)

SPECIAL FUNCTIONS

See also 4524a-b, 4637, 4727, 4741.

4663:

✱Jarden, Dov. Recurring sequences: a collection of papers. Riveon Lematematika, Jerusalem, 1958. ii+101 pp. (mimeographed) $5.00.

Twenty-six articles, with some revision and consolidation, by Jarden (some with co-authors A. Katz, Th. Motzkin, and Z. Tuchman), published since 1946 in Amer. Math. Monthly, Math. Student, and Riveon Lematematika, concerned with the properties of Fibonacci and related sequences. Those originally in Hebrew are here in English. Included is a revision of the table in Riveon Lematematika 11 (1957), 70–90 [MR 20 #3643] and of the bibliography in Riveon Lematematika 2 (1948), 36–45 [MR 9, 423].

E. G. Straus (Los Angeles, Calif.)

4664:

Moser, L.; and Wyman, M. Asymptotic development of the Stirling numbers of the first kind. J. London Math. Soc. 33 (1958), 133–146.

The Stirling numbers of the first kind S_n^m are defined by $x(x-1)\cdots(x-n+1)=\sum_{m=1}^n S_n^m x^m$, $1 \leq m \leq n$. Hammersley [Proc. London Math. Soc. (3) 1 (1951), 435–452; MR 13, 725] has raised the problem of obtaining approximate formulas for S_n^m, and has given one formula valid for a restricted range of m. Similarly, C. Jordan gives the formula $|S_n^m| \sim (n-1)!(\log n+\gamma)^{m-1}/(m-1)!$, valid for m fixed and n large. In this paper the authors solve the problem for large n by deriving three complete asymptotic formulas valid, respectively, in the ranges (1) $m=o(\log n)$; (2) $m\to\infty$, $m \leq n-O(n^\alpha)$, α fixed, $0<\alpha<1$; (3) $n-o(n^{\frac{1}{2}}) \leq m \leq n$. They also give a check formula for the range (2') $m/\log n \to \infty$, $m \leq n-O(n^\alpha)$. The complete results are too complicated to give here, but we can quote the following. The first term in range (1) is Jordan's formula, and the first term in range (3) is $(-1)^{n+m}\binom{n}{m}(m/2)^{n-m}$. Illustrative numerical results are given for S_{50}^m for various values of m.

N. J. Fine (Princeton, N.J.)

4665:

✱Rey Pastor, J.; and de Castro Brzezicki, A. Funciones de Bessel. [Bessel functions.] Teoría Matemática y Aplicaciones a la Ciencia y a la Técnica. Editorial Dossat, S. A., Madrid, 1958. xii+240 pp. 180 ptas.

This is a treatise of the first quality on the modern theory of Bessel functions, of interest to both the theoretical and applied mathematician as well as to the physicist and the technologist. A great abundance of fine examples taken from science and technology appear for the first time in this text.

The authors have made good selections in topics which are of interest to those who wish to apply Bessel functions, and they also present a unified theory of the subject. There are many sketches and graphs and throughout the text many exercises in which, in general, new theorems are presented. At the end of each chapter there are notes and additions with detailed bibliographical reviews of the various subjects.

There are 13 chapters on the following topics. Chapter I: the differential equation and the functions of Bessel, and applications. In it are defined the functions $J_0(x)$, $J_1(x)$, the Bessel differential equation of order n, Bessel functions of the first kind of any real order whatsoever, the general integral of Bessel's equation and functions of the second kind, Bessel functions of the third kind (Hankel functions), and the hyperbolic Bessel functions $I_n(x)$. Chapter II: the coefficients and the integral of Bessel. The generation of the functions $J_n(x)$, following Schlömilch, is described, as well as bounds for the $J_n(x)$, hyperbolic functions of Bessel of the first kind and of integral order, the Bessel integral, some developments in Fourier series in which the functions $J_n(x)$ occur, the

development of x^n in terms of Bessel coefficients, differential equations satisfied by $J_n(x)$ and $I_n(x)$, along with recurrence formulas for Bessel functions, the formula of Parseval, Lommel's definition of $J_n(x)$, and formulas for the numerical computation of the $J_n(x)$. Chapter III: the fundamental linear relations, cylinder functions, Bessel functions of order $\pm\frac{1}{2}$, the spherical functions of Bessel, with physical applications of the functions of Stokes, Lamb, etc., Thomsons's functions ber x and bei x and analogous functions, the functions of Struve, the functions of Anger-Weber, Lommel, and the functions her, hei, yer, yei. Chapter IV: differential equations which can be integrated by Bessel functions, types A, B, and C, and more general equations, differential equations which appear in various physical applications, hypergeometric equations of Legendre and Bessel, and the functions of Airy. Chapter V: integrals involving Bessel functions, including the discontinuous integral of Weber and Schafheitlin. Chapter VI: integral representations of Bessel functions, the method of Laplace, modification of the contours. Chapter VII: theorems of addition of Bessel functions, the theorem of Neumann-Graf, and the formulas of Gegenbauer. Chapter VIII: asymptotic developments of Bessel functions. Chapter IX: existence, reality and separation of zeros. Chapter X: series of Neumann and of Kapteyn, along with Lommel functions, integral representations and integrals of Fresnel, and the polynomials of Gegenbauer. Chapter XI: series of Schlömlich, of Fourier-Bessel, and of Dini, and the Gibbs phenomenon. Chapter XII: integral representation of arbitrary functions, the theorem of Fourier, theorems of Cherry, and other transformation formulas. Chapter XIII: further applications, e.g. concerning the Laplace equation, the equation of diffusion of heat, the wave equation, the equation of Tricomi, the equation of Schrödinger, the theory of hereditary and teleological oscillations, and statistics and probability.

Appendix I consists of a table of differential equations with integrals in terms of Bessel functions, the study of which led the authors to a practical rule, given in Chapter IV, for the determination of the integrability of such equations by means of Bessel functions, and for the formation of their general integrals. Appendix II gives tables of Bessel functions and a bibliography.

This excellent book will be useful both for orientation in the theory and for reference in further work.

E. Frank (Chicago, Ill.)

4666:

Krakowski, M. On certain functions connected with the Bessel functions. Zastos. Mat. **4** (1958), 130–141. (Polish. Russian and English summaries)

By simple computations the function $M_p(z)=\int_0^\infty e^{-z\cosh t}\sinh pt\,dt$ occurring in calculations of impedance is expressed as a product of $\exp(-z)$ and a polynomial in z^{-1} and p, for integer values of p. It is shown that $M_p(z)$ is a special solution of a certain non-homogeneous Bessel differential equation. For integer values of p an asymptotic behavior of $G_p(z)=K_p(z)+M_p(z)$, where $K_p(z)$ is Bessel function of the third kind (with imaginary argument), is pointed out and the graphs of real and imaginary parts of G_{-1}, G_0, G_1 are given. Some definite integrals are expressed in closed form using G_n.

C. Masaitis (Havre de Grace, Md.)

4667:

Fukuhara, Masuo; and Ohashi, Saburo. On a P-function expressible by elementary functions. Sûgaku **8** (1956/57), 27–29. (Japanese)

In the former paper [Sûgaku **2** (1950), 227–230] the authors proved that Riemann's P-function $y=\begin{Bmatrix} a & b & c & \\ \lambda & \mu & \nu & x \\ \lambda' & \mu' & \nu' & \end{Bmatrix}$ can be expressed by elementary functions, only in the following special cases (where $\alpha=\lambda'-\lambda$, $\beta=\mu'-\mu$, $\gamma=\nu'-\nu$): 1° $\alpha+\beta+\gamma$ odd; 2° $\alpha=\frac{1}{2}+p$, $\beta=\frac{1}{2}+q$; 3° $\alpha=\frac{1}{2}+p$, $\beta=\frac{1}{3}+q$, $\gamma=\frac{1}{3}+r$; 4° $\alpha=\frac{1}{2}+p$, $\beta=\frac{1}{3}+q$, $\gamma=\frac{1}{4}+r$; 5° $\alpha=\frac{1}{3}+p$, $\beta=\frac{1}{3}+q$, $\gamma=\frac{1}{3}+r$; 6° $\alpha=\frac{1}{3}+p$, $\beta=\frac{1}{4}+q$, $\gamma=\frac{1}{4}+r$; p, q, r are arbitrary integers, except that in 6° their sum must be odd. In this paper, the authors obtain the explicit forms of elementary functions in these special cases.

M. Tsuji (Tokyo)

ORDINARY DIFFERENTIAL EQUATIONS

See also 4655, 4666, 5019, 5030.

4668:

Hayashi, Kyuzo. On transformations of differential equations whose second members are discontinuous. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **29** (1955), 131–137.

Consider a vector differential system $y'=f(x, y)$, where $f(x, y)$ is defined on L: $0\leqq x\leqq a$ and R^n: $|y|<\infty$. Assume that $f(x, y)$ has the properties needed for the Carathéodory existence theorem. After transforming the dependent variables, the author extends the domain of definition of the coefficient functions to $L\times S^n$, the segment times the n-sphere. The compactness of S^n can then be used to discuss the extension of the solution curves. Compare the author's earlier paper [same Mem. **28** (1954), 313–325; MR **16**, 472].

L. Markus (Minneapolis, Minn.)

4669:

Hayashi, Kyuzo. On quasi-equicontinuous sets. Sets of solutions of a differential equation. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **31** (1958), 9–23.

In this paper the author examines an abstract formulation of the transformation problem for differential equations, which he analysed in his paper reviewed above. He first studies the maps from a topological space into a locally compact uniform space, with special reference to the Alexander compactification. He then applies his results to the mappings of differential equations as stated in the above review.

L. Markus (Minneapolis, Minn.)

4670:

Avdeev, N. Ya. Construction of an analytic function defined by a differential equation. Rostov. Gos. Ped. Inst. Uč. Zap. **4** (1957), 69–74. (Russian)

4671:

Parodi, Maurice. Détermination de courbes planes définies par une inégalité entre les valeurs absolues de fonctions des éléments de contact en un point courant. C. R. Acad. Sci. Paris **245** (1957), 1871.

Let ϕ_1, ϕ_2 and ϕ_3 be three functions of (x, y, y') such that the differential equation

$$\begin{vmatrix} \phi_1 & \phi_2 & \phi_3 \\ \phi_2 & \phi_1 & \phi_3 \\ \phi_2 & \phi_3 & \phi_1 \end{vmatrix}=0$$

has a solution y. Then, by the theorem of Hadamard, the same function y yields a solution of the differential inequality $|\phi_1|\leqq|\phi_2|+|\phi_3|$. {Since differential inequalities are a valuable aid in the study of nonlinear partial differential equations, the result may have applications in that field.} *R. M. Redheffer* (Los Angeles, Calif.)

4672:

Parodi, Maurice. Remarque au sujet d'un travail antérieur. C. R. Acad. Sci. Paris **246** (1958), 3573–3574.

By a theorem of Ostrowski concerning matrices, the solutions y of the differential equation in the preceding review satisfy not only $|\phi_1|\leqq|\phi_2|+|\phi_3|$, but certain other inequalities involving a parameter α.

R. M. Redheffer (Los Angeles, Calif.)

4673:

Duffin, R. J.; and Serbyn, W. D. Approximate solution of differential equations by a variational method. J. Math. Phys. **37** (1958), 162–168.

The authors obtain a lower bound for the solution of the differential equation $(pu')'-qu=0$, positive at $t=a$ and $t=b$, provided that the associated quadratic functional $\int_a^b[pu'^2+qu^2]dt$ is positive definite. As an application of this result, lower bounds are obtained for integrals of the form $\int_x^\infty e^{g(s)}ds$, and an excellent approximation is obtained in this way for the integral $\int_x^\infty e^{-s^2}ds$.

R. Bellman (Santa Monica, Calif.)

4674:

Glazman, I. M. On the negative part of the spectrum of one-dimensional and multi-dimensional differential operators on vector-functions. Dokl. Akad. Nauk SSSR (N.S.) **119** (1958), 421–424. (Russian)

A statement without proofs of several results concerning the boundedness and L^2-behavior of solutions of equations of the form $(-1)^n y^{(2n)}+A(x)y=0$, where y is an N-dimensional vector and $A(x)$ is an hermitian matrix. Analogous results are given for the partial differential vector equation $\Delta u=A(p)u$.

R. Bellman (Santa Monica, Calif.)

4675:

Barrett, John H. Second order complex differential equations with a real independent variable. Pacific J. Math. **8** (1958), 187–200.

In the interval $[a,\infty)$ let $q(x)=q_1(x)+iq_2(x)=r(x)\times\exp[i\theta(x)]$, where $r(x)$ is continuous and $\theta(x)$ is of class C'. Let $s=s[a,x;q]$ and $c=c[a,x;q]$ be the solutions of

(1) $$(y'/q)'+\bar{q}y=0$$

with the initial conditions $s(a)=0$, $s'(a)=q(a)$, $c(a)=1$, $c'(a)=0$. If $q(x)$ is real, then

$$s=\sin\{\textstyle\int_a^x q(t)dt\},\quad c=\cos\{\textstyle\int_a^x q(t)dt\}.$$

If $q_2(x)\not\equiv0$, one still has $|s|^2+|c|^2=1$, but many other properties of the trigonometric functions are lost. If $b(x)\equiv\frac{1}{2}\theta'(x)/r(x)$ is of class C' and $b'(x)\neq0$, then $s[a,x;q]$ has no zeros in $[a,\infty)$. If $b(x)=\text{const.}$, the functions $s(x)$ and $c(x)$ can be found explicitly. In this case $s[a,x;q]$ has infinitely many zeros in $[a,\infty)$ if and only if $\int_a^\infty r(t)dt=\infty$, and, if $b\neq0$, $c[a,x;q]$ has no zeros. A general differential equation of the form $[p(x)y']'+f(x)y=0$ can always be reduced to the special form (1) with the aid of a complex Prüfer transformation. *E. Hille* (New Haven, Conn.)

4676:

Perčinkova-V'čkova, Danica. Sur un problème de Sturm-Liouville. Fac. Philos. Univ. Skopje. Sect. Sci. Nat. Annuaire **9** (1956), 31–36. (Serbo-Croatian. French summary)

On dresse le tableau des valeurs caractéristiques et des fonctions caractéristiques du problème suivant:

$$y'''+\lambda y=0$$
$$y^{(p)}(a)=y^{(q)}(a)=y^{(r)}(b)=0.$$

Du résumé de l'auteur

4677:

Corduneănu, C. Solutions asymptotiquement presque-périodiques des équations différentielles non-linéaires du second ordre. Acad. R. P. Romîne. Fil. Iaşi. Stud. Cerc. Şti. **6** (1955), no. 3–4, 1–4. (Romanian. Russian and French summaries)

Si $f(x,y,z)$ est asymptotiquement presque-périodique et vérifie les conditions de la note précédente [i.e., Com. Acad. R. P. Romîne **5** (1955), 793–797; MR **17**, 39] seulement pour $x\geqq0$, toutes les solutions bornées pour $x\geqq0$ sont asymptotiquement presque-périodiques. La démonstration est la même que celle de la proposition précédente. *I. Barbălat* (Zbl **67**, 66)

4678:

Yoshizawa, Taro. Note on the solutions of a system of differential equations. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **29** (1955), 249–273.

Consider $y''=f(x,y,y')$, where the real function $f(x,y,y')$ is defined in a domain $\mathscr{D}$: $a\leqq x\leqq b$, $\underline{\omega}(x)\leqq y\leqq\bar{\omega}(x)$, $-\infty<y'<\infty$, where it is continuous in (y,y') and measurable in x. Also assume $|f(x,y,y')|\leqq M(x)$ for $|y'|\leqq L$, a given bound, and $M(x)$ a summable function. Under further technical restrictions on $\underline{\omega}(x)$, $\bar{\omega}(x)$, and $f(x,y,y')$ (too difficult to state here), the author proves the existence of a solution through two boundary points.

The author also considers the vector differential system $\dot{x}=F(t,x)$ in $0\leqq t<\infty$, $-\infty<x_i<\infty$ $(i=1,2,\cdots,n)$, with $F(t,x)$ continuous in x and measurable in t, and $|f(t,x)|\leqq M(t)$, a summable function in $0\leqq t\leqq T$, when $|x|\leqq L$. He proves certain theorems on the ultimate boundedness of the solutions and applies his results to the second order equation $\ddot{x}+f(x,\dot{x},t)\dot{x}+g(x)=e(t)$ under appropriate conditions on the coefficients. The required hypotheses are quite complicated. *L. Markus* (Minneapolis, Minn.)

4679:

Yoshizawa, Taro. Note on the boundedness and the ultimate boundedness of solutions of $x'=F(t,x)$. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **29** (1955), 275–291.

Consider the vector differential system $\dot{x}=F(t,x)$, where $F(t,x)$ is continuous for $0\leqq t<\infty$ and all x. The author defines several types of boundedness (for $t\to+\infty$) for the solutions: a) The solution $x=x(t;x_0,t_0)$ is said to be (simply) bounded if $|x(t;x_0,t_0)|<B(x_0,t_0)$ for $t\geqq t_0$; b) the solutions issuing from $t=t_0$ are equi-bounded if for $|x_0|\leqq\alpha$, the bound $B=B(\alpha,t_0)$ is independent of the particular solution; c) the solutions are uniformly bounded if $B=B(\alpha)$ is independent of t_0.

The following theorem is typical of the results of this paper. Theorem: Assume $F(t,x)$ is periodic in t. If the solutions issuing from $t=0$ are equi-bounded and the solutions issuing from $t>0$ are simply bounded, then they are uniformly bounded. *L. Markus* (Minneapolis, Minn.)

4680:

Yoshizawa, Taro. On the necessary and sufficient condition for the uniform boundedness of solutions of $x'=F(t,x)$. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **30** (1957), 217–226.

The author proves converse statements of theorems on boundedness, ultimate boundedness, etc., of solutions which appeared in a previous paper [#4679 above].

J. L. Massera (Montevideo)

4681:

Yoshizawa, Taro. Appendix to the paper "Note on the boundedness and the ultimate boundedness". Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **30** (1957), 91–103.

Complements to a previous paper [#4679 above]. The solutions of $\dot{x}=F(t,x)$, $(t,x)\in[0,\infty)\times E_n$, are said to be totally bounded if given $\alpha>0$ there exist $\beta,\gamma>0$ such that $\|x_0\|\leqq\alpha$, $\|H(t,x)\|\leqq\gamma$ for all $\alpha<\|x\|<\beta$, $t\geqq 0$, imply that the solution of $\dot{x}=F+H$ through (t_0,x_0) satisfies $\|x\|<\beta$ for $t\geqq t_0$. The following are some of the results proved in the present paper. 1. If (1) is linear, total boundedness implies uniform ultimate boundedness and that all the solutions tend to zero as $t\to\infty$. 2. If F is periodic in t and has continuous partials with respect to x, and if the solutions are equibounded and ultimately bounded, they are totally bounded. 3. If a Lyapunov function φ exists which satisfies the assumptions of theorem 4 of the above mentioned paper and if φ satisfies a local Lipschitz condition with respect to x, uniformly with respect to t, the solutions are totally bounded.

J. L. Massera (Montevideo)

4682:

Reid, William T. Principal solutions of non-oscillatory self-adjoint linear differential systems. Pacific J. Math. **8** (1958), 147–169.

Consider the system of linear differential equations (1) $u'=A(x)u+B(x)v$, $v'=C(x)u-A^*(x)v$ for n-vectors u, v (or for n by r matrices $u=U$, $v=V$), where $A(x)$, $B(x)$, $C(x)$ are continuous n by n matrices for $x>0$ such that $B=B^*$ is positive definite, $C=C^*$, and an asterisk denotes Hermitian transposition. The system (1) is termed non-oscillatory on an x-interval if every solution vector $(u,v)\not\equiv 0$ has at most one zero on the interval. If $(u,v)=(U_1,V_1)$, (U_2,V_2) is a pair of matrix solutions of (1), then $\{U_1,U_2\}\equiv U_1^*V_2-V_1^*U_2$ is a constant matrix. Let (U,V) be an n by n matrix solution of (1) for which $\det U\neq 0$ on some x-interval. Let $K=\{U,U\}$, $D(x)=-U^{-1}BU^{*-1}$, $T(x)=T(x,s,U)$ the solution of $T'=DKT$ and $T(s)=E$, and $S(x,s,U)=-\int_s^x T^{-1}(t,s,U)D(t)dt$. An n by n matrix solution (U,V) of (1) is called principal if $\det U\neq 0$ for large x and $S^{-1}(x,s,U)\to 0$ as $x\to\infty$ for one (and/or all large) s. Some of the main results can be described as follows: If (1) is non-oscillatory for $x>0$ and if $0<s<t$, then (1) has an n by n matrix solution $(U_{st}(x),V_{st}(x))$ such that $U_{st}(x)$ is E or 0 according as x is s or t; $(U_s(x),V_s(x))=\lim(U_{st}(x),V_{st}(x))$, as $t\to\infty$, exists for $x>0$, $\det U_s(x)\neq 0$, $\{U_s,U_s\}=0$ (so that $T(x,r,U)\equiv E$) and (U_s,V_s) is a principal solution of (1); any other principal solution is of the form (U_sK^0,V_sK^0), where K^0 is a non-singular constant matrix. Furthermore, if (U,V) is any n by n matrix solution such that $\det U\neq 0$ for large x, then $S^{-1}(t,s,U)\to-\{U,U_s\}U(s)$ as $t\to\infty$, while $\{U,U_s\}$ is non-singular if and only if $U^{-1}(x)U_s(x)\to 0$ as $x\to\infty$. This extends results of the reviewer who considered a less general system of differential equations and solutions (U,V) satisfying $\{U,U\}=0$ [Duke Math. J. **24** (1957), 25–35; MR **18**, 576].

P. Hartman (Baltimore, Md.)

4683:

Morris, G. R. A differential equation for undamped forced non-linear oscillations. II. Proc. Cambridge Philos. Soc. **54** (1958), 426–438.

In a previous paper [same Proc. **51** (1955), 297–312; MR **16**, 1026] the author studied the solutions of 1) $\ddot{x}+2x^3=e(t)$ where $e(t)$ is continuous, even, of least period 2π. He displayed an infinite family of solutions $x_k(t)$, for k an integer greater than some fixed number p, with period 2π. Here $\dot{x}_k(0)=0$ and $x_k(0)=a_k$.

In this paper he studies solutions of 1) with period $2m\pi$, for integers $m>1$. The principal result is the following theorem. If $k>p$, then for each $m>1$ there are at least $\phi(m)$ solutions of 1) with minimal period $2m\pi$ and amplitude on the closed segment $[a_k,a_{k+1}]$. Here $\phi(m)$ is the number of positive integers less than m and prime to m.

L. Markus (Minneapolis, Minn.)

4684:

Vinograd, R. È. The general case of the stability of characteristic exponents and the existence of leading coordinates. Dokl. Akad. Nauk SSSR (N.S.) **119** (1958), 633–635. (Russian)

A statement without proof of various results concerning the asymptotic behavior of linear equations of the form $x'=P(t)x$, under particular assumptions concerning the form of $P(t)$.

R. Bellman (Santa Monica, Calif.)

4685:

Lefschetz, Solomon. Liapunov and stability in dynamical systems. Bol. Soc. Mat. Mexicana (2) **3** (1958), 25–39.

This paper is a study of stability, particularly of Liapunov's second method, as it appears in the framework of topological dynamics. The author first defines and discusses stability, asymptotic stability, and uniform stability for the vector systems $\dot{x}=X(t,x)$ and $\dot{x}=X(x)$ in Euclidean n-space. He defines Liapunov functions, and using them states the four general theorems on stability due to A. M. Liapunov which constitute what is known as Liapunov's second method.

Next, the fundamentals of dynamical systems from topological dynamics are summarized. The notation is as in Chapter V of Nemyckiĭ and Stepanov, "Qualitative theory of differential equations" [OGIZ, Moscow-Leningrad, 1947; MR **10**, 612]. The author emphasizes particularly the results in Zubov, "The methods of A. M. Liapunov and their application" [Izdat. Leningrad. Univ. Moscow, 1957; MR **19**, 275]. The various types of stability for a closed invariant set in a dynamical system are defined. Such a set is here assumed to have a neighborhood with compact closure.

The author gives conditions for stability of closed invariant sets. For example: (I) A necessary and sufficient condition for the stability of a closed invariant set M is that, given $\varepsilon>0$, we can find $\xi>0$ such that if a point p is at a distance greater than ε from M, then no point of the path through p is within a distance ξ of M for any negative t; (II) a necessary and sufficient condition that M be asymptotically stable is the condition of (I) together with the existence of a neighborhood of M free from complete paths other than those paths in M itself.

Removing the requirement that M have a compact neighborhood, the author defines a "generalized Liapunov function" attached to M, and obtains theorems analogous to the four theorems of Liapunov mentioned earlier. Theorems on uniformity of stability are also obtained. The abstract results are illustrated with several examples from Euclidean space.

W. S. Loud (Minneapolis, Minn.)

4686:

Pozharitskiĭ, G. K. On the construction of the Liapunov functions from the integrals of the equations for perturbed motion. J Appl. Math. Mech. **22** (1958), 203–214 (145–154 Prikl. Mat. Meh.).

This is essentially an algebraic discussion of how we

may seek a combination $\phi(U_1, \cdots, U_p)$ of functions $U_1, \cdots, U_p$ of variables $x_1, \cdots, x_n, t$ which shall — if possible — be positive definite with respect to $x_1, \cdots, x_n$; it is assumed that each of $U_1, \cdots, U_p$ is a power series in $x_1, \cdots, x_n$ vanishing with these arguments, and the procedure is, of course, to eliminate — if possible — the linear terms. *T. M. Cherry* (Victoria)

4687:

Roitenberg, Ia. N. A method for the construction of Liapunov functions for linear systems with variable coefficients. J. Appl. Math. Mech. 22 (1958), 230–236 (167–172 Prikl. Mat. Meh.).

The method is to choose a differential-equation system with constant coefficients which in some sense approximates the given system, and try whether a Liapunov function for the former is, in some domain, a Liapunov function for the latter also. *T. M. Cherry* (Melbourne)

4688:

Barocio, Samuel. On trajectories in the vicinity of a three-dimensional singularity. Bol. Soc. Mat. Mexicana (2) 1 (1956), 57–58. (Spanish)

The author gives an example of a three-dimensional vector field with a single singularity, O. There exist two spheres, S_1 and S_2 with center O, S_1 interior to S_2, with the following properties. Every trajectory of the field which intersects S_2 enters S_2. No trajectory which intersects S_2 ever intersects S_1, and hence every trajectory which intersects S_2 remains at a positive distance from O. *W. S. Loud* (Minneapolis, Minn.)

4689:

Parsons, D. H. One-dimensional diffusion, with the diffusion coefficient a non-linear function of concentration. J. London Math. Soc. 33 (1958), 246–251.

The initial value problem for the one-dimensional diffusion equation with semi-constant initial data is studied in cases when the diffusion coefficient D is related to the concentration c by the formulae $D=(\alpha+\beta c)^n$ or $D=k\exp(\alpha+\beta c)$ (α, β, k, n constants). The problem is reduced to the consideration of $(DE)p'+up+2(m+u^2)p^2+u(1+2m+u^2)p^3=0$, $m=1/n$, for $p(u)$ on $(-\infty, \infty)$. The methods and results are similar to those in author's previous account for the case $n=1$ [Quart. Appl. Math. 15 (1957), 298–303; MR 19, 917]. *I. I. Kolodner* (Albuquerque, N.M.)

4690:

Pliss, V. A. Eiserman's problem in the case of three simultaneous differential equations. Dokl. Akad. Nauk SSSR 121 (1958), 422–425. (Russian)

The author presents a number of conditions under which all the solutions of a nonlinear system of the form $dx/dt=Ax+f(x_1)b$, $x(0)=c$ approach zero as t approaches infinity, where x is an n-dimensional vector, A is a constant matrix, b is a constant vector, and $f(x_1)$ is a scalar function of one component of x, say x_1. *R. Bellman* (Santa Monica, Calif.)

4691:

Norkin, S. B. On periodic solutions of a linear homogeneous differential equation of second order with retarded argument. Mat. Sb. N.S. 45(87) (1958), 71–104. (Russian)

The author presents a detailed investigation of the problem of determining periodic solutions of the functional equation $u''(x)+2au'(x)+bu(x)+q(x)u(x-t(x))=0$, the type of equation so extensively investigated by A. Myškis. The basic technique is that of successive approximations. *R. Bellman* (Santa Monica, Calif.)

4692:

Perov, A. I. On integral inequalities. Voronež. Gos. Univ. Trudy Sem. Funkcional. Anal. 1957, no. 5, 87–97. (Russian)

The author develops certain relations between solutions of ordinary integral equations and the corresponding integral inequalities; then he applies these results to uniqueness theory of ordinary differential equations $du/dt=f(u, t)$ even for the case with $u(t)$ a function with values in a normed linear space. *M. M. Day* (Urbana, Ill.)

4693:

Foyaš, Č. [Foiaș, C.]; Gussi, G.; and Poenaru, V. Generalized solutions of a quasilinear differential equation in Banach space. Dokl. Akad. Nauk SSSR (N.S.) 119 (1958), 884–887. (Russian)

Let X be a Banach space and $A(t)$, $t\in(a, b)$, a linear closed operator satisfying the following hypothesis (I): there exists a sequence $\{J_n\}$ of idempotent operators such that (i) $J_nJ_m=0$ for $n\neq m$; (ii) $I=\bigcup_1^\infty J_n$; (iii) $J_nA(t)\subseteq A(t)J_n$; (iv) $A(t)J_n$ is bounded and continuous in norm in $X_{(n)}=J_n(X)$, $n=1, 2, \cdots$ [cf. the authors' paper, Trans. Amer. Math. Soc. 86 (1957), 335–347; MR 19, 1185]. Let $\mathfrak{X}$ be the space of sequences $\{x_n\}$, where $x_n\in X_{(n)}$, with seminorm $\sup(\|x_{n1}\|, \cdots, \|x_{nk}\|)$, and $\tilde{X}$ the space of sequences $\{J_nx\}$. We consider the equations (1) $dx/dt=A(t)x(t)$ in X and (1′) $dx/dt=\tilde{A}(t)x(t)$. Under (I) the Cauchy problem for (1′) is correctly set in $\mathfrak{X}$ and has a solution which is unique in (a, b) for arbitrary initial data in $\mathfrak{X}$; if $x(t)$ is the solution of the Cauchy problem for (1) with $x(c)=x_0$, then $\tilde{x}(t)$ is the solution of the Cauchy problem for (1′) with $\tilde{x}(c)=\tilde{x}_0$.

The preceding theorem, as well as others of a similar orientation, is announced without proof. The others are obtained by replacing (I) with other sets of conditions, by admitting Sobolev solutions, and finally by considering the equation $dx/dt=A(t)x(t)+f(t, x)$. *R. N. Goss* (San Diego, Calif.)

PARTIAL DIFFERENTIAL EQUATIONS

See also 4572, 4674, 4689, 4738, 4783, 4875, 4944, 5024, 5035.

4694:

Bielecki, A.; et Kisyński, J. Sur un problème de Mlle Z. Szmydt relatif à l'équation $\partial^2z/\partial x\partial y=f(x, y, z, \partial z/\partial x, \partial z/\partial y)$. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. 6 (1958), 321–325.

Let $\varphi(u)$ be a continuous, non-decreasing function for $u\geqq 0$ satisfying the Osgood condition $\int_{+0} du/\varphi(u)=\infty$. Let $f(x, y, z, p, q)$, $G(x, z, q)$, $H(y, z, p)$, $g(x)$, $h(y)$ be continuous, bounded functions for $|x|\leqq\alpha$, $|y|\leqq\beta$ and arbitrary z, p, q. Let $|f(x, y, z, p_2, q_2)-f(x, y, z, p_1, q_1)|\leqq\varphi(|p_2-p_1|+|q_2-q_1|)$ and let G, H, g, h satisfy uniform Lipschitz conditions with respect to q, p, x, y, respectively, with Lipschitz constants A, B, a, b satisfying $AB<1$, $ab<1$. Let $|g|\leqq\beta$, $|h|\leqq\alpha$, $g(0)=h(0)=0$, $|\xi|\leqq\alpha$, $|\eta|\leqq\beta$, ζ arbitrary. Then $z_{xy}=f(x, y, z, z_x, z_y)$ has at least one C^1-solution possessing a continuous mixed derivative $z_{xy}=z_{yx}$ on $|x|\leqq\alpha$, $|y|\leqq\beta$ and satisfying $z_x=G(x, z, z_y)$ for $y=g(x)$, $z_y=H(x, z, z_x)$ for $x=h(y)$ and $z(\xi, \eta)=\zeta$. The proof is of a standard type

using Schauder's fixed point theorem [cf., e.g., Szymdt, Bull. Acad. Polon. Sci. Cl. III 4 (1956), 579–584; MR 18, 741], but depends on an ingenious choice of an invariant convex set under the functional mapping employed.

P. Hartman (Baltimore, Md.)

4695:

Gutmann, M. Sur l'ensemble des champs biscalaires. Bul. Inst. Politehn. București 18 (1956), no. 3–4, 109–117. (Romanian. Russian and French summaries)

The author discusses the Pfaff problem by direct methods for a vector field which determines ∞^1 surfaces in three dimensions [for the general theory, see J. A. Schouten, "Der Ricci-Kalkul", Springer, Berlin, 1924; p. 119]. First, the author considers a vector field which is a scalar multiple of a gradient. By use of the integrability condition, it is shown that the sum of two such fields leads to ∞^1 surfaces. Finally, the case where the integrability condition is not identically satisfied is discussed.

N. Coburn (Ann Arbor, Mich.)

4696:

Višik, M. I.; and Lyusternik, L. A. Asymptotic theory of the solutions of problems involving rapidly oscillating boundary conditions for partial differential equations. Dokl. Akad. Nauk SSSR (N.S.) 119 (1958), 636–639. (Russian)

The authors adapt their method of solution of the first boundary-value problem for the elliptic equation (*) $L_{2k}u=0$ to boundary conditions which become rapidly oscillating near the boundary Γ; this method is based on the construction of a sequence of approximations to u in the neighborhood of a boundary point [same Dokl. 113 (1957), 734–737; MR 19, 861; and in more detail, Uspehi Mat. Nauk (N.S.) 12 (1957), no. 5(77), 3–122; MR 20 #2539]. A family of functions $\{f_\varepsilon\}$ depending on a parameter ε is said to be $(1/\varepsilon)$-oscillating in the interval $\mu(\varphi_0\leqq\varphi\leqq\varphi_1)$ of Γ if, for any φ in this interval and a K not depending on ε, (**) $|\int_{\varphi_0}^{\varphi} f_\varepsilon(\varphi)d\varphi|<K\varepsilon$. The family is $(1/\varepsilon)$-oscillating on Γ if Γ can be covered by a finite number of intervals μ_i in each of which it is $(1/\varepsilon)$-oscillating. The method cited yields almost immediately an asymptotic solution u_ε of (*) which satisfies the conditions $\partial^l u_\varepsilon/\partial\rho^l=\alpha_l(\varphi)g_{l\varepsilon}(\varphi)$ on Γ ($l=0, 1, \cdots, k-1$), where φ is the arc parameter on Γ, ρ is the distance along the normal to Γ, $\alpha_l(\varphi)\equiv0$ outside one of the intervals μ_i, and $g_{l\varepsilon}(\varphi)$ are $(1/\varepsilon)$-oscillating functions on μ_i. Estimates in terms of ε are obtained for the magnitudes of the approximations to u and its derivatives and for the error term. Corresponding results are stated for functions which are $(1/\varepsilon)$-oscillating of order s, the definition here being essentially the same as that above with ε^s replacing ε in the right member of (**).

R. N. Goss (San Diego, Calif.)

4697:

Pettineo, Benedetto. Sul problema di derivata obliqua per le equazioni lineari a derivate parziali del secondo ordine di tipo ellittico in due variabili. Atti Accad. Sci. Lett. Arti Palermo. Parte I (4) 16 (1955/56), 5–26 (1957).

Consider the elliptic equation

(1) $$Eu\equiv\partial^2u/\partial x_1{}^2+\partial^2u/\partial x_2{}^2+\sum b_h\partial u/\partial x_h+c'u=f(x)$$

in a planar bounded domain D, and the boundary condition

(2) $$A\partial u/\partial n+A_1\partial u/\partial\sigma+\beta u=q$$

on its boundary FD. Assume that: (i) b_h, c', f, q, β, A_1, A belong to C^λ $(0<\lambda<1)$ and FD belongs to $C^{1+\lambda}$; and (ii) the system $E(u)-\omega u=0$, $A\partial u/\partial n-A_1\partial u/\partial\sigma+\beta_0u=0$ has no nontrivial solutions for some β_0 of class C^λ and a number ω satisfying $c'-\omega<0$. The system (1), (2) is then reduced to a Fredholm integral equation. In particular, uniqueness for (1), (2) implies existence.

A. Friedman (Berkeley, Calif.)

4698:

Pettineo, Benedetto. Sulle equazioni integrali singolari nel piano. Atti Accad. Sci. Lett. Arti Palermo. Parte I (4) 16 (1955/56), 35–59 (1957).

Continuing his previous work [see the preceding review] the author proves that the restriction (ii) can be weakened and replaced by the conditions

$$\int_D \Psi(x)q_1(x)dx+\int_{FD}\Phi(x)q_2(x)d\sigma(x)=0,$$

where q_1, q_2 depend only on f, q, and Ψ, Φ run over the eigensolutions of a certain system of Fredholm integral equations.

A. Friedman (Berkeley, Calif.)

4699:

Pettineo, Benedetto. Sulla funzione di Green pel problema di Dirichlet relativo alle equazioni lineari ellittiche. Atti Accad. Sci. Lett. Arti Palermo. Parte I (4) 16 (1955/56), 65–68 (1957).

Continuing his previous work [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 20 (1956), 306–311; MR 18, 741] the author uses a series expansion for Green's function (which converges in the L^2 sense) to get a uniformly convergent series expansion for any solution of the Dirichlet problem which vanishes on the boundary.

A. Friedman (Berkeley, Calif.)

4700:

Weston, Vaughan H. Solutions of the Helmholtz equation for a class of non-separable cylindrical and rotational coordinate systems. Quart. Appl. Math. 15 (1958), 420–427.

In cylindrical and rotational coordinate systems, one of the variables is separated out of the Helmholtz equation $(\Delta+k^2)\psi=0$, leaving a second order partial differential equation in two variables. If the metric coefficients of the transformation formulae satisfy certain relations, it is shown that for rotational coordinates (u_1, u_2, φ) solutions exist of the form

$$\psi=e^{i\mu\varphi}B(u_2)\sum_r a_r(u_1)[h_\varphi(u_1, u_2)]^r,$$

and for cylindrical coordinates (u_1, u_2, z),

$$\psi=e^{i\mu z}B(u_2)\sum_r a_r(u_1)\left[\begin{matrix}x\\y\end{matrix}(u_1,u_2)\right]^r.$$

The coefficients $a_r(u)$ can be found from a recurrence set of ordinary differential equations.

An application of the method is given by the author in same Quart. 16 (1958), 237–257.

D. J. Hofsommer (Amsterdam)

4701:

Osserman, Robert. On the inequality $\Delta u\geqq f(u)$. Pacific J. Math. 7 (1957), 1641–1647.

The inequality of the title is studied in n variables by an ingenious method which is more elementary and more powerful than that used in the previous work of Wittich, Haviland and Walter. Theorem 1 states the following. Let $f(t)$ be positive, continuous, and monotone increasing for $t\geqq t_0$, and suppose

$$\int^\infty\left(\int_0^t f(s)ds\right)^{-\frac{1}{2}}dt<\infty.$$

Then a twice continuously differentiable function u

cannot satisfy $\Delta u>0$ throughout space and $\Delta u \geqq f(u)$ outside of some sphere S. The method also allows more detailed statements in interesting special cases; for example, if u satisfies $\Delta u \geqq \varepsilon e^{2u}$ for $r \leqq R$ and $u(0)=a$, then $R \leqq 2\varepsilon^{-\frac{1}{2}}e^{-a}$.

As an application the author establishes the following. If a simply-connected surface S has a Riemannian metric whose Gauss curvature K satisfies $K \leqq -\varepsilon < 0$ everywhere, then S is conformally equivalent to the interior of the unit circle. For related work see Keller, Comm. Pure Appl. Math. **10** (1957), 503–510 [MR **19**, 964], and Calabi, Duke Math. J. **25** (1957), 45–56 [MR **19**, 1056]. {The author refers to a result of the reviewer, in which for $n=2$, the hypothesis that $f(t)$ be increasing is shown to be superfluous. [See: A general maximum-modulus theorem, NAR Report, Jan. 1956, Theorem XXV.] The reviewer wishes to point out that the hypothesis of his theorem should be supplemented by $\Delta u \geqq 0$, to rule out entire solutions bounded above.}

R. M. Redheffer (Los Angeles, Calif.)

4702:

Itô, Seizô. Fundamental solutions of parabolic differential equations and boundary value problems. Jap. J. Math. **27** (1957), 55–102.

Let D be a domain of an m-dimensional orientable C^∞ manifold M such that the boundary ∂D consists of at most enumerable components each of which is a C^3 hypersurface. Let $L_{tx}=(A_{tx}-a(t,x)^{-\frac{1}{2}}\partial a(t,x)^{\frac{1}{2}}/\partial t)-\partial/\partial t$ be a parabolic differential operator (i.e.

$$A_{tx}=a(t,x)^{-\frac{1}{2}}\frac{\partial}{\partial x^i}\left(a^{ij}(t,x)a(t,x)^{\frac{1}{2}}\frac{\partial}{\partial x^j}\right)$$
$$-a(t,x)^{-\frac{1}{2}}\frac{\partial}{\partial x^i}\left(b^i(t,x)a(t,x)^{\frac{1}{2}}\right)+c(t,x)$$

is, for $-\infty \leqq s_0 < t < t_0 \leqq \infty$, elliptic in $\bar{D}$) such that $\partial a^{ij}/\partial t$, $\partial^2 a^{ij}/\partial x^k \partial x^l$, $\partial b^i/\partial x^k$ and c are uniformly Hölder-continuous on $(s_0, t_0)\times\bar{D}$ and that there exists a finite valued function $c_0(s,t)$ satisfying $-\infty < c(\tau,x) \leqq c_0(s,t)$ for any $\langle\tau,x\rangle \in (s,t]\times\bar{D}$. The author proves the existence of the fundamental solution of L_{tx} pertaining to a fairly general mixed-type boundary condition. It is an extension of the author's construction [Duke Math. J. **24** (1957), 299–312; MR **19**, 864] for the case where $\bar{D}$ is compact; and it is based upon the observation that the fundamental solution is monotone increasing when the domain is enlarged appropriately assigned with a suitable boundary condition. The fundamental solution for L_{tx} is ingeniously used to show the existence of a Green function of an elliptic differential operator with a mixed-type boundary condition. These results, combined with semi-group theory and the Hellinger-Hahn spectral analysis, enable the author to derive anew the formula of eigen-function expansions for elliptic differential operators with boundary conditions. *K. Yosida* (Tokyo)

4703:

Devingtal', Yu. V. On the existence of a solution to a Frankl' problem. Dokl. Akad. Nauk SSSR (N.S.) **119** (1958), 15–18. (Russian)

The Frankl' problem for the equation (*) $\operatorname{sgn} y\cdot|y|^m u_{xx}+u_{yy}=0$ $(m>0)$ is reduced to the consideration of a Fredholm integral equation. In this problem one seeks a solution of (*) in a region D bounded by the segment joining $A'(0,-1)$ to $A(0,1)$, a smooth curve Γ in the upper halfplane joining A to $B(a,0)$, the characteristic of (*) which joins A' to $C(b,0)$, and the segment CB. Here $a>b=2/(m+2)$. The solution is to satisfy the boundary conditions: $u=\psi_1(s)$ on Γ, $u=\psi_2(s)$ on CB, $u_x=0$ on AA', and $u(0,y)-u(0,-y)=f(y)$ on AA'. The problem has been solved for somewhat less general equations than (*) by A. V. Bicadze [same Dokl. **109** (1956), 1091–1094; **112** (1957), 375–376; MR **18**, 743; **19**, 861].

The region D is replaced by a new region D_1 obtained by adjoining to D the region symmetric to D with respect to the y-axis. Along the x-axis additional conditions $u(x,0)=\tau(x)$, $u_y(x,0)=\nu(x)$ are imposed, and the resulting Cauchy problem in the lower (hyperbolic) halfplane is solved. The given boundary conditions enable $\nu(x)$ to be eliminated and the solution of (*) in the region D_1^+ in the elliptic half-plane to be handled as a Dirichlet problem. This in turn is reduced to a Fredholm integral equation by means of the Green's function for D_1^+, the solution of which is equivalent to the solution of the given Frankl' problem. *R. N. Goss* (San Diego, Calif.)

4704:

Gronec, Yu. [Hronec, J.] Die normale Form der partiellen Differentialgleichungen zweiter Ordnung mit n Variabeln. Acta Fac. Nat. Univ. Comenian. Math. **2** (1958), 165–175. (Russian. Slovak and German summaries)

The author employs standard results concerning quadratic forms to reduce the general second order linear differential equation to various canonical forms.

R. Bellman (Santa Monica, Calif.)

4705:

Pogodin, Yu. Ya.; Sučkov, V. A.; and Yanenko, N. N. On progressive waves of the equations of gas dynamics. Dokl. Akad. Nauk SSSR (N.S.) **119** (1958), 443–445. (Russian)

As in the earlier paper of the third author [same Dokl. **109** (1956), 44–47; MR **18**, 214] a solution $u_i(x_1,\cdot\cdot,x_m)$ of the system of m homogeneous first order linear equations in m dependent variables u_i and m independent variables x_k which satisfies $m-r$ functional relations $\varphi_\alpha(u_1,\cdots,u_m)=0$, is called a progressive wave. Here such a wave is said to be of rank r. As an example of progressive waves of rank $m-1$, the authors consider the equations of a polytropic gas under both adiabatic and isothermal conditions. Treating the case $m=3$ in detail, they develop an explicit condition for the progressive wave to have the freedom of two arbitrary functions of a single argument. Finally they illustrate the formation of progressive waves of rank 1 and of rank 2 by solving the problem of motion of a polytropic gas induced by two semi-infinite plane pistons moving independently at right angles to one another.

R. N. Goss (San Diego, Calif.)

4706:

Krahn, Dorothee. On the iterated wave equation. IA, IB. Nederl. Akad. Wetensch. Proc. Ser. A **60**=Indag. Math. **19** (1957), 492–505.

Cet article est consacré à l'étude du problème de Cauchy pour l'équation

$$\text{(A)} \qquad \left(\Delta-\frac{\partial^2}{\partial t^2}\right)^n u=0,\ \Delta=\sum_{i=1}^m \frac{\partial^2}{\partial x_i^2}$$

et sa généralisation

$$\left(a_1^2\frac{\partial^2}{\partial t^2}-\Delta\right)\cdots\left(a_n^2\frac{\partial^2}{\partial t^2}-\Delta\right)u=0.$$

L'auteur établit une formule qui permet de représenter toute solution de l'équation (A) en combinaison linéaire convenable d'expressions de la forme

$t^p u^{(-k)}$ $(k=0, 2, \cdots, 2(n-1))$,

$u^{(-k)}$ étant solution de $\Delta+(k/t)\partial/\partial t-\partial^2/\partial t^2$.

Grace à ce résultat, l'auteur ramène la résolution des problèmes de Cauchy qui l'intéressent aux problèmes correspondants pour l'équation d'Euler-Poisson-Darboux, pour lesquels la solution est connue. *H. P. Garnir* (Liège)

4707:

Antohi-Teodorescu, Veronica. La solution élémentaire de l'équation aux dérivées partielles du quatrième ordre à caractéristique double. II. An. Univ. "C.I. Parhon" Bucureşti. Ser. Şti. Nat. **7** (1958), no. 17, 9–21. (Romanian. Russian and French summaries)

In an earlier paper [same An. **6** (1957), no. 15, 9–24; MR **20** #3377] the author studied a fourth-order partial differential equation in m-dimensional Riemann space when m is an odd number. In the present paper the case of an even m is discussed. As might be expected, the expansion of an elementary solution in powers of Γ now contains logarithmic terms. If $m\leqq 4$, the singularity is purely logarithmic, while one part of the elementary solution has a pole and another part a logarithmic singularity if $m>4$. *A. Erdélyi* (Pasadena, Calif.)

4708:

Melamed, E. Ya. On stability of the solutions of some partial differential boundary problems in Banach space. Dokl. Akad. Nauk SSSR **120** (1958), 1194–1195. (Russian)

This paper contains statements without proof of various results concerning the stability of solutions of linear partial differential equations of the form $L(u)=A(x, t)u$, where L is one of the usual operators, and $A(x, t)$ is a general operator function satisfying various conditions.

R. Bellman (Santa Monica, Calif.)

DIFFERENTIAL ALGEBRA

4709:

Goldman, Lawrence. Lowest order equation for zeros of a homogeneous linear differential polynomial. Illinois J. Math. **2** (1958), 567–576.

Let $\mathcal{F}$ be an ordinary differential field of characteristic zero with algebraically closed constant field C, $L_n(y)\in\mathcal{F}\{y\}$ a homogeneous linear differential polynomial of order n, and $x_1, \cdots, x_n$ a fundamental system of zeros of $L_n(y)$ such that the constant field of $\mathcal{F}\langle x_1, \cdots, x_n\rangle$ is C. Relations are obtained between $L_n(y)$, or the Picard-Vessiot group G of differential automorphisms of $\mathcal{F}\langle x_1, \cdots, x_n\rangle$ over $\mathcal{F}$, and the lowest equation of a zero x of $L_n(y)$, i.e., the lowest order irreducible differential polynomial in $\mathcal{F}\{y\}$ annulling x. Typical results are: If the lowest equation of x over $\mathcal{F}$ is homogeneous of order r, there exist $L_r(y)$, $L_{n-r}(y)$ in $\mathcal{F}\langle x\rangle\{y\}$ such that $L_n(y)=L_{n-r}(L_r(y))$ and $L_r(x)=0$. Each zero $x\neq 0$ of $L_n(y)$ is of order n over $\mathcal{F}$ if and only if G operates transitively on the zeros of $L_n(y)$. If $G=\mathrm{O}_n(C)$, $n>1$, there exists an irreducible homogeneous differential polynomial $Q(y)\in\mathcal{F}\{y\}$ of degree 2 and order $n-1$ such that, for each zero $x\neq 0$ of $L_n(y)$, $Q(x)\in C$ and $Q(y)-Q(x)$ is the lowest equation of x over $\mathcal{F}$. *M. Rosenlicht* (Evanston, Ill.)

POTENTIAL THEORY

See also 4641, 4642, 4915, 4952.

4710:

Gegelia, T. G. The fundamental lemma of I. I. Privalov for space potentials. Soobšč. Akad. Nauk Gruzin. SSR **18** (1957), 257–264. (Russian)

For Q and R points in 3-space, with Q lying on a given surface S, and $M(Q, R)$ and $\varphi(Q)$ functions measurable in Q, let

$$\Phi(R)=\iint_S \frac{M(Q, R)\varphi(Q)}{|Q-R|^2}\,dS_Q;\ \Gamma(R)=\iint_S \frac{M(Q, R)}{|Q-R|^2}\,dS_Q.$$

It is assumed further that $M(Q, R)$ is bounded, that

$$|M(Q,\ R_1)-M(Q,\ R_2)|\leq\frac{K|R_1-R_2|}{|Q-R_1|+|Q-R_2|}\quad (K=\text{const.}),$$

that the singular integral $\Gamma(P)$ exists at $P\in S$, that S admits a tangent plane at P, and that the angular limits of $\Gamma(R)$ exist at P (in the sense of approach through cones with vertex at P and not elsewhere touching the tangent plane). Theorem: Under the above conditions the existence of the singular integral $\Phi(P)$ is necessary and sufficient for the existence of the angular limits of $\Phi(R)$ at P; moreover, the jump condition $\Phi^{\pm}(P)-\Phi(P)=[\Gamma^{\pm}(P)-\Gamma(P)]\varphi(P)$ is then valid. The above result represents a spatial analogue of the lemma of Privalov for Cauchy-Lebesgue integrals in the plane.

M. G. Arsove (Seattle, Wash.)

4711a:

Šaškin, Yu. A. On uniqueness in the inverse problem of potential theory. Dokl. Akad. Nauk SSSR **115** (1957), 64–66. (Russian)

4711b:

Todorov, I. T.; and Zidarov, D. Uniqueness of the determination of the shape of an attracting body from the values of its external potential. Dokl. Akad. Nauk SSSR **120** (1958), 262–264. (Russian)

P. C. Novikov has called the following problem the "inverse problem of potential theory" [Dokl. Akad. Nauk SSSR (N.S.) **18** (1938), 165–168]: given a positive mass distribution μ on a bounded region G, to determine G from the knowledge of the potential of μ in the neighborhood of infinity. The sweeping-out process shows that the problem has no unique solution, in general. However, if μ has density 1 on G, and if G is star-shaped with respect to the origin and has a smooth boundary, then G is uniquely determined by the exterior potential of μ [Novikov, loc. cit.].

Both of the present papers furnish other conditions on G sufficient to ensure its unique determination by the exterior potential of μ (assumed to have strictly positive density on G). *M. G. Arsove* (Seattle, Wash.)

4712:

Jackson, Lloyd K. Subfunctions and the Dirichlet problem. Pacific J. Math. **8** (1958), 243–255.

In previous papers [Beckenbach and Jackson, same J. **3** (1953), 291–313; MR **14**, 1084; Jackson, ibid. **5** (1955), 215–228; MR **16**, 1108] the notion of subharmonic function was generalized by replacing the dominating family of harmonic functions with a more general family of functions. The object was to require of the dominating functions the minimum or most tractable properties

needed for studying boundary-value problems by subfunction techniques. These properties can be separated naturally into two sorts: those that yield functions that are solutions in the interior and those that assure agreement with prescribed boundary values. The present paper is concerned primarily with properties of the first sort.

Results are obtained for the linear equation $\Delta z+a(x, y)z_x+b(x, y)z_y+c(x, y)z=f(x, y)$ and the quasi-linear equation $a(p, q)r+2b(p, q)s+c(p, q)t=0$, where $\Delta z=\partial^2z/\partial x^2+\partial^2z/\partial y^2$ and $p=\partial z/\partial x, \cdots, t=\partial^2z/\partial y^2$. In particular, a result is given ensuring the existence and uniqueness of the solution of the Dirichlet problem for the minimal-surface equation $(1+q^2)r-2pqs+(1+p^2)t=0$ for a nonconvex region under suitable restrictions.

E. F. Beckenbach (Los Angeles, Calif.)

FINITE DIFFERENCES AND FUNCTIONAL EQUATIONS

See also 4519, 4631, 4663, 4691.

4713:

Baumol, William J. Topology of second order linear difference equations with constant coefficients. Econometrica **26** (1958), 258–285.

The phase plane topology of second order linear difference equations with constant coefficients is investigated in great detail. In this instance the phase plane is one in which suceeding values of the dependent variable (for unit increment of the independent variable) are taken as abscissa and ordinate in a rectangular coordinate system. The results are applied to Hicks' trade cycle model.

E. Pinney (Berkeley, Calif.)

4714:

Hahn, Wolfgang. Über geometrische Differenzengleichungen von unendlicher Ordnung. Math. Nachr. **18** (1958), 19–35.

The author studies the linear geometric difference equation of infinite order: 1) $Lf\equiv\sum_{r=0}^{\infty}(B_r-A_rx)f(q^rx)=0$, in which x is a complex number, q satisfies the inequality $0<q<1$, and the complex constant coefficients A_r, B_r are such that both the series $\sum_{r=0}^{\infty}A_rz^r\equiv g(z)$ and $\sum_{r=0}^{\infty}B_rz^r\equiv h(z)$ converge in some $|z|<R$. A solution is sought in the form of an integral: 2) $f(x)=\int_C x^tF(t)dt$. It is found that C must be an appropriately chosen path of integration and that $F(t)$ must be an analytic solution of the difference equation $F(t+1)=H(q^t)F(t)$, in which the characteristic function $H(z)$ is defined by $H(z)=g(z)/h(qz)$. If the integral converges, a series representation may be obtained in the form 3) $f(x)=x^{\gamma}\sum_j x^jR_j(\gamma)$ in either ascending or descending powers of x, depending on the nature of the poles of the integrand. Further, a necessary and sufficient condition that equation 1) possess solutions of form 3) is that the characteristic function $H(z)$ have poles. To each such pole corresponds one such solution. Any finite number of such solutions are linearly independent in the q-difference sense. The series part of 3) constitutes a generalized Heine series.

Equation 1) is recast as a q-difference equation of infinite order in the q-differences and the limiting forms as $q\to1$ of the resulting equation and its solutions are related respectively to differential equations of infinite order and generalizations of the hypergeometric series. Examples with particular characteristic functions are discussed. It is demonstrated that the theory of finite order Heine series may be developed from 1) by the integral solution method 2) for $H(z)$ an appropriate rational function.

P. E. Guenther (Cleveland, Ohio)

4715:

Wing, G. Milton. Solution of a time-dependent, one-dimensional neutron transport problem. J. Math. Mech. **7** (1958), 757–766.

By means of the theory of invariant imbedding [Bellman, Kalaba and Wing, Proc. Nat. Acad. Sci. U.S.A. **43** (1958), 517–520; MR **19**, 506], functional equations of novel type are obtained for the description of one-dimensional neutron transport processes, and for more general neutron processes as well. In this paper, the author establishes the existence and uniqueness of the solution of the nonlinear equation

$$u_x+2u_t/c=\lambda^{-1}\int_0^1 v(x, t)u(x, t-z)dz+\lambda^{-1},\quad x\geqq0,\ t\geqq0,$$

where $v(x, t)=u_t$, subject to the boundary conditions $v(x, 0)=0$, $v(0, t)=0$. The method used is one of successive approximations. The study of the asymptotic behavior necessitates the derivation of various results concerning the zeros of exponential polynomials.

R. Bellman (Santa Monica, Calif.)

4716:

Chaundy, T. W.; and McLeod, J. B. On a functional equation. Quart. J. Math. Oxford Ser. (2) **9** (1958), 202–206.

A problem in the statistical thermodynamics of mixtures leads the authors to consider the functional equation

$$(1)\qquad f(x)+uf(vx)=Uf(Vx),$$

where x, u, v are independent parameters, $f(x)$ is the continuous unknown function, and $U(u, v)$, $V(u, v)$ are unknown functions dependent on $f(x)$. The thermodynamics problem requires consideration only of positive values of x, u, v, U, V, but in the mathematical analysis x is taken positive, u and U real, while v and V are possibly complex.

By an ingenious sequence of formal manipulations the authors prove that the form of the solution depends on a quadratic characteristic equation. If this equation has distinct roots, the general solution of 1) is shown to be $f(x)=Ax^a+Bx^b$, where A, B, a, b are constants and U, V are determined by the conditions $1+uv^a=UV^a$, $1+uv^b=UV^b$. If the roots of the characteristic equation are equal, the form of the solution is $f(x)=(A+B\log x)x^a$, where A, B, a are constants and U, V are determined by the conditions $1+uv^a=UV^a$, $u\log v=U\log V$. In the special case $B=0$ the solution reduces to $f(x)=Ax^a$, with u, v, U, V satisfying the single condition $1+uv^a=UV^a$. The case $A=B=0$, $f(x)=0$ is dismissed as sufficiently pointless.

P. E. Guenther (Cleveland, Ohio)

SEQUENCES, SERIES, SUMMABILITY

See also 4788.

4717:

Berekašvili, V. A. Euler and Borel summation for double series. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze **24** (1957), 53–69. (Russian)

The (E, q) transform of the sequence S_{mn} $(m, n=0, 1, 2, \cdots)$ is defined by

$$E_{mn}=(q+1)^{-m-n}\sum_i\sum_j\binom{m}{i}\binom{n}{j}q^{m+n-i-j}S_{ij}\quad(q>0).$$

The sequence S_{mn} is (E, q) summable to S if $E_{mn}\to S$ as m and $n\to\infty$. The sequence S_{mn} is (E, q, λ) summable to S if $E_{mn}\to S$ when m and n tend to ∞ under the restriction that $1/\lambda\leq m/n\leq\lambda$ $(\lambda>1)$.

The paper gives several consistency and inclusion results for these methods of summation and for several others. A typical result is: If $\lim_{m,n\to\infty} S_{mn}=S$, and if $|S_{mn}|<A\exp\{o(m)+o(n)\}$, then the sequence $\{S_{mn}\}$ is (E, q, λ) summable to S. *W. H. J. Fuchs* (Ithaca, N.Y.)

4718:

Petersen, G. M. Matrix norms. Quart. J. Math. Oxford Ser. (2) **9** (1958), 161–168.

Four lemmas imply the following theorem. For each $r=1, 2, \cdots$, let $A^{(r)}$ be a regular matrix with elements $a_{mn}^{(r)}$ for which $a_{mn}^{(r)}\to 0$ as $m, n\to\infty$. Let $\sum_{n=0}^{\infty}|a_{mn}^{(r)}|\leq M$ for each m and r. Suppose finally that each bounded sequence evaluable $A^{(r-1)}$ is also evaluable $A^{(r)}$. Then there exist regular matrices $B^{(r)}$ and $D^{(r)}$ such that $\sum_{n=1}^{\infty}|b_{mn}^{(r)}|\leq M$, each bounded sequence evaluable by one of the two methods $A^{(r)}$ and $B^{(r)}$ is also evaluable by the other, and $B^{(r)}=D^{(r)}B^{(r-1)}$. The other result is free of the restriction that sequences be bounded. Let $B_1, B_2, \cdots$ be the regular matrices defined by $B_k=A^kC$, where A is the matrix of the transformation $t_n=2s_{2n}-s_{2n+1}$ and C is the Cesàro matrix of order 1. Then the elements $b_{mn}^{(k)}$ of B_k satisfy the condition

$$\lim_{m\to\infty}\sum_{n=0}^{\infty}|b_{mn}^{(k)}|=1$$

for each k, and B_{k+1} includes B_k for each k, but there is no regular matrix D which includes B_k for each k. {The paper contains errata. The main theorem credited to Broudno is due to Mazur and Orlicz; and exposition of this matter is given by Mazur and Orlicz [Studia Math. **14** (1954), 129–160; MR **16**, 814].}

R. P. Agnew (Ithaca, N.Y.)

4719:

Olevskiĭ, A. M. On linear methods of summation. Dokl. Akad. Nauk SSSR **120** (1958), 701–703. (Russian)

Announcement of results of the following type. Theorem 1: Let $\mathscr{A}$ be a countable set of Toeplitz (summability) matrices. Let $\sum u_n(x)$ be a series with bounded terms divergent for every x in a set E. By suitable insertion of zeros between the u_n a series can be produced which is not summable by any matrix in $\mathscr{A}$ for any $x\in E$. Theorem 2: Let $\{C_n\}$ $(n=1, 2, \cdots)$ be a set of regular, row-finite Toeplitz matrices, $A_n=C_nC_{n-1}\cdots C_1A_0$, A_0 a Toeplitz matrix. Then there is a matrix method B stronger than all the methods A_j and such that any sequence summable to $+\infty$ by A_j is summed to $+\infty$ by B.

W. H. J. Fuchs (Ithaca, N.Y.)

4720:

Reĭmers, È. G. Mean value theorems and multiplication of summable series. Dokl. Akad. Nauk SSSR **120** (1958), 1196–1199. (Russian)

Various conditions (some of them necessary and sufficient) are stated under which summability of $\sum u_k$ to U by the triangular matrix method A and summability of $\sum v_k$ to V by the triangular matrix method B imply summability of $\sum_n\sum_{k\leq n}u_kv_{n-k}$ to UV by the triangular matrix method C. This is a continuation of work by W. Jurkat and A. Peyerimhoff [Math. Z. **55** (1951), 92–108; MR 13, 934]. *W. H. J. Fuchs* (Ithaca, N.Y.)

4721:

Hsu, L. C. Concerning the condition of uniform boundedness for a type of scalar-to-vector transformations. J. Indian Math. Soc. (N.S.) **21** (1957), 115–126.

The author considers a generalization of the Toeplitz theorem on summability in which the transformation does not carry sequences to sequences but carries a family of real-valued functions on $(0, \infty)$ to a space of X-valued functions on $(0, \infty)$, where X is a finite-dimensional normed space. The proof uses the general summability theorems of the reviewer [Trans. Amer. Math. Soc. **51** (1942), 583–608; MR 4, 14].

M. M. Day (Urbana, Ill.)

4722:

Rajagopal, C. T. A Tauberian theorem for the Riemann-Liouville integral of integer order. Canad. J. Math. **9** (1957), 487–499.

Suppose $s(x)$ is integrable in every finite interval $(0, a)$, and that for $\lambda>1$, $\limsup_{t\to\infty}\sup_{t\leq t'\leq\lambda t}((s(t')-s(t))/t^{q-p})=W(\lambda)$ exists (finitely). Then, equivalently, this limit exists finitely if the t in the denominator is replaced by t'; denote it by $W^*(\lambda)$. Let $s_p(x)$, p a positive integer, be the pth integral of $s(x)$, i.e. $(\Gamma(p))^{-1}\int_0^x(x-t)^{p-1}s(t)dt$. Let A and B be the sums of the negative and positive terms, respectively, of $-\sum_{\nu=0}^{p}(-1)^{\nu}\binom{p}{\nu}\{1+(p-\nu)(\lambda-1)/p\}^q$ and C and D the sums of the negative and positive terms, respectively, of $\sum_{\nu=0}^{p}(-1)^{\nu}\binom{p}{\nu}\{1-\nu(1-\theta)/p\}^q$. Clearly, A and B depend on λ, p, q; C and D depend on θ, p, q.

The main result of this paper is

$$-\left(\frac{\lambda-1}{p}\right)^p\liminf_{x\to\infty}\frac{s(x)}{x^{q-p}}\leq A\liminf_{x\to\infty}\frac{s_p(x)}{x^q}+B\limsup_{x\to\infty}\frac{s_p(x)}{x^q}+\left(\frac{\lambda-1}{p}\right)^{p-1}\int_{1+(1-p^{-1})(\lambda-1)}^{\lambda}W(t)dt,$$

and for $0<\theta<1$,

$$\left(\frac{1-\theta}{p}\right)^p\limsup_{x\to\infty}\frac{s(x)}{x^{q-p}}\leq C\liminf_{x\to\infty}\frac{s_p(x)}{x^q}+D\limsup_{x\to\infty}\frac{s_p(x)}{x^q}+\left(\frac{1-\theta}{p}\right)^{p-1}\int_{\theta}^{1-(1-p^{-1})(1-\theta)}W^*(t)dt.$$

The result may be further elaborated if one requires, in addition, that the limits used to define W and W^* still exist if "lim sup sup" is replaced by "lim inf inf".

A number of known tauberian theorems and some extensions of these theorems are shown to be consequences of this result. The following is typical. Let $s(x)$ be an integral and $s'(x)=O_R(x^{q-p-1})$ as $x\to\infty$ through almost all $x\geq 0$, p a positive integer, q real. Then $s_p(x)/x^q\to L$ $(x\to\infty)$ implies $s(x)/x^{q-p}\to Lq(q-1)\cdots(q-p+1)$.

D. Waterman (Lafayette, Ind.)

4723:

Šalát, Tibor. Absolutely convergent series. Mat.-Fyz. Časopis. Slovensk. Akad. Vied **7** (1957), 139–142. (Slovak. Russian summary)

Generalizing a result of J. Jakubík [same Časopis **5** (1955), 133–136; MR **17**, 728] the author shows that the set W of elements $w=\sum c_ia_i$ (where $c_i\in C_i$; $i=1, 2, \cdots$) is perfect (and compact) under the following conditions: The a_i are fixed elements taken from a Banach space X; the C_i are non-empty sets of complex numbers with the following properties: (1) All C_i are compact; (2) $\sum K_i\|a_i\|<\infty$, where $K_i=\sup_{z\in C_i}|z|$; (3) an infinite number of sets C_i contain more than one element. Proof: The usual

diagonalization process shows that W is compact. Because of (3) (and (2)) no point in W is isolated. W is, of course, not empty.
K. Zeller (Tübingen)

4724:

Arrighi, Gino. Considerazioni sulla serie $\sum_1^\infty s\, P_s(t)e^{\rho_s t}$. Boll. Un. Mat. Ital. (3) 13 (1958), 38–45.

Series of this type appear in the definition of functions of matrices previously given by the author [Ann. Scuola Norm. Sup. Pisa (3) 8 (1954), 141–156; MR 17, 252]. In this paper a number of sufficient conditions for convergence are established.
W. Ledermann (Manchester)

APPROXIMATIONS AND EXPANSIONS

See also 4745.

4725:

Balázs, J.; and Turán, P. Notes on interpolation. III. Convergence. Acta Math. Acad. Sci. Hungar. 9 (1958), 195–214.

This investigation is a contribution to "lacunary" interpolation. If the interpolation points are $x_\nu = x_{\nu n}$, $\nu = 1, 2, \cdots, n$, we form the polynomials

$$R_n(x) = \sum_{\nu=1}^{n} \alpha_\nu r_\nu(x) + \sum_{\nu=1}^{n} \beta_\nu \rho_\nu(x)$$

of degree $2n-1$ for which $R_n(x_\nu) = \alpha_\nu$ and $R_n''(x_\nu) = \beta_\nu$ are prescribed. The "fundamental polynomials" $r_\nu(x)$, $\rho_\nu(x)$ are uniquely determined. The authors investigate the particular case when the $x_{\nu n}$ are the zeros of $(1-x^2) \times P_{n-1}'(x)$, P_n Legendre's polynomial, and study the convergence of $R_n(x)$, $n \to \infty$, associated with a given function $f(x)$, $\alpha_\nu = f(x_{\nu n})$, β_ν given constants. The following two principal results are obtained. Let $f'(x)$ be continuous with modulus $\omega(\delta)$ such that $\int t^{-1}\omega(t)dt < \infty$. Let $n^{-1}\max_{(\nu)} |\beta_{\nu n}| \to 0$ as $n \to \infty$. Then $R_n(x) \to f(x)$ uniformly in $-1 \leqq x \leqq 1$. Let $0 < \varepsilon < 1$ be given; there exists $F(x) \in \mathrm{Lip}(1-\varepsilon)$ such that the associated polynomials $R_n(x)$ (even for $\beta_{\nu n} = 0$) are unbounded for $x = 0$.
G. Szegö (Stanford, Calif.)

4726:

Berman, D. L. The distribution of nodes in the Bernstein interpolation process. Dokl. Akad. Nauk SSSR 119 (1958), 1063–1065. (Russian)

The author has given [same Dokl. 60 (1948), 333–336; MR 9, 584] sufficient conditions for the points $-1 \leqq x_1^{(n)} < \cdots < x_n^{(n)} \leqq 1$, $n = 1, 2, \cdots$, in order that (*) $A_n(f, x) \to f(x)$ uniformly on $[-1, +1]$ for each continuous f, where $A_n(f, x)$ is the Bernstein interpolation polynomial for a fixed natural number p and the nodes $x_k^{(n)} = x_k$. These conditions are $|l_k(1)| \leqq |l_{k+1}(1)|$, $|l_k(-1)| \geqq |l_{k+1}(-1)|$, $k = 1, 2, \cdots, n-1$, where $l_k(x)$ are the fundamental Lagrange polynomials of the nodes x_k. He now gives a geometric interpretation of these conditions in terms of the "conjugate" points $X_k^{(n)} = x_k + (x_{k+1} - x_k) \times (1-a_k)^{-1}$, $a_k = -\omega_n'(x_k)/\omega_n'(x_{k+1})$, $\omega_n(x) = \prod_{j=1}^n (x - x_j)$. He also gives necessary conditions for (*): $C_1 n^{-1} \leqq \theta_k^{(n)} - \theta_{k+1}^{(n)} \leqq C_2 n^{-1}$, where $\theta_k^{(n)}$ is defined by $\cos \theta_k^{(n)} = x_k^{(n)}$.
G. G. Lorentz (Syracuse, N.Y.)

4727:

Campbell, R. Sur les polynomes orthogonaux dont les dérivés sont orthogonaux. Monatsh. Math. 61 (1957), 143–146.

L'Autore dà un procedimento per determinare le famiglie di polinomi ortogonali con derivate ortogonali, servendosi della relazione di ricorrenza stabilita da Favard [C.R. Acad. Sci. Paris 200 (1935), 2052–2053] per i polinomi ortogonali. Scritta tale relazione pei polinomi e per le loro derivate, si arriva, con opportune eliminazioni, ad un sistema di equazioni alle differenze nei coefficienti delle due relazioni scritte. La risoluzione di tale sistema dà luogo a due casi: uno conduce ai polinomi L_n^α di Laguerre, l'altro ai polinomi $J_n^{\alpha,\beta}$ di Jacobi.
L. Gori (Rome)

4728:

Lee, Wen-ching. On the degree of approximation by Bernstein polynomials. Advancement in Math. 4 (1958), 567–568. (Chinese)

For a real continuous function f on $[0, 1]$, let ω be its modulus of continuity, and B_n the nth Bernstein polynomial of f. The known inequality $|B_n(x) - f(x)| \leqq \frac{5}{4}\omega(n^{-\frac{1}{2}})$ [cf. G. G. Lorentz, Bernstein polynomials, Univ. of Toronto Press, 1953; MR 15, 217] is slightly sharpened by replacing the constant $\frac{5}{4}$ by $\frac{19}{16}$.
Ky Fan (Notre Dame, Ind.)

FOURIER ANALYSIS

See also 4617, 4639, 4745.

4729:

Bononcini, Vittorio E. Sulle serie di Fourier delle funzioni composte. Rend. Sem. Mat. Univ. Padova 27 (1957), 218–227.

In a previous work, M. Pagni discussed the problem of expressing the Fourier coefficients of $f(g(t))$ in terms of those of f and g. The author establishes the required representation theorems under weaker conditions.
R. Bellman (Santa Monica, Calif.)

4730:

Chow, H. H. On Borel summability of the Fourier series of a function with an infinite limit. Acta Math. Sinica 6 (1956), 472–475. (Chinese. English summary)

Let $f(\theta)$ be Lebesgue-integrable and of period 2π; further let

$$\varphi(t) = \tfrac{1}{2}\{f(\theta_0 + t) + f(\theta_0 - t)\}.$$

If, as $t \to 0+$, $\varphi(t)$ increases monotonically to $+\infty$, then the Fourier series of $f(\theta)$ is at $\theta = \theta_0$ Borel-summable to the sum $+\infty$; i.e.,

$$\lim_{n \to +\infty} \frac{2}{\pi} \int_0^\pi \frac{\varphi(t)}{t} e^{-x(1-\cos t)} \sin(x \sin t)dt = +\infty.$$

K. Mahler (Manchester)

4731:

Lee, Ching-Hsi. On the Gibbs phenomenon for the Riemann summation $(R, 1)$ of Fourier series. Acta Math. Sinica 6 (1956), 418–425. (Chinese. English summary)

The series $\sum_{n=1}^\infty u_n$ is said to be summable $(R, 1)$ to s if (i) $\sum_{n=1}^\infty u_n(\sin nh)/nh$ converges for all $h \neq 0$, and (ii) $\lim_{h \to 0} \sum_{n=1}^\infty u_n(\sin nh)/nh = s$. Let

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^\infty (a_n \cos nx + b_n \sin nx) = A_0(x) + \sum_{n=1}^\infty A_n(x)$$

be the Fourier series of $f(x)$ and let

$$R_h(x) = A_0(x) + \sum_{n=1}^\infty A_n(x) \frac{\sin nh}{nh}.$$

The Riemann-Gibbs set of $\{R_h(x)\}$ at x_0 is defined as the

set of all values $\lambda=\lim_{h\to 0} R_h(x_0+\alpha(h))$ where $\lim_{h\to 0}\alpha(h)=0$, $\lim_{h\to 0}(\alpha(h)/h)=\beta$ $(-\infty\leqq\beta\leqq+\infty)$. The author proves that the Riemann-Gibbs set of $\{R_h(x)\}$ for the series $\sum_{n=1}^{\infty} (\sin nx)/n$ at $x=0$ is the closed interval $[-\frac{1}{2}\pi, +\frac{1}{2}\pi]$. If further $f(x)$ is of bounded variation and has a jump at $x=\xi$, then the Riemann-Gibbs set of $\{R_h(x)\}$ at ξ is the closed interval of length $|f(x+0)-f(x-0)|$ with centre at $\frac{1}{2}\{f(x+0)+f(x-0)\}$. The second result depends on the uniform convergence of $\{R_h(x)\}$ at every point of continuity of the function $f(x)$ of bounded variation.

K. Mahler (Manchester)

4732:

Flett, T. M. On the degree of approximation to a function by the Cesàro means of its Fourier series. Quart. J. Math. Oxford Ser. (2) 7 (1956), 81–95.

Let $f(x)$ be an integrable function in the interval $(-\pi, \pi)$ and $\sigma_n^\alpha(x)$ be the nth Cesàro mean of order α of the Fourier series of f. Let $\varphi_x(t)=f(x+t)+f(x-t)-2f(x)$. It is known that if $0<\alpha<1$, $0<\delta<\pi$, $k>\alpha$, and x is a point such that $\varphi_x(t)=O(t^\alpha)$ for $0\leqq t\leqq\delta$, then (*) $\sigma_n^k(x)-f(x)=O(n^{-\alpha})$. The author treats the case $k\leqq\alpha$ and proves the following theorems. (i) If $0<\alpha<1$, $0<\delta<\pi$, and x is a point such that (**) $\int_0^t |d\varphi_x(u)|=O(t^\alpha)$, then (*) holds with $k=\alpha$. (ii) The order of the integral of (**) in (i) cannot be weakened. (iii) If $0<\alpha<1$, $0\leqq\beta<1$, $0<\delta\leqq\pi$, $k\geqq\alpha-\beta$, and x is a point such that nth term of the Fourier series of f is of order $n^{-\beta}$ at the point x and (**) holds when $0\leqq t\leqq\delta$, then (*) holds. (iv) If $\alpha=1$ in (iii), then $\sigma_n^k(x)-f(x)=O(n^{-1}\log n)$ for $k\geqq 1-\beta$.

S. Izumi (Sapporo)

4733:

Kennedy, P. B. Fourier series with gaps. Quart. J. Math. Oxford Ser. (2) 7 (1956), 224–230.

Let (n_k) be an increasing sequence of integers such that $n_{k+1}-n_k\to\infty$ $(k\to\infty)$. Let $f(x)$ be an integrable function in the interval $(0, 2\pi)$ and its Fourier coefficients a_n, b_n be zero for $n\neq n_k$ $(k=1, 2, \cdots)$. The author proves the following theorems. (i) If $f(x)$ is of bounded variation in some subinterval of $(0, 2\pi)$, then a_n, $b_n=O(1/n)$ $(n\to\infty)$. (ii) If $f(x)$ belongs to a Lipschitz class Lip α $(0<\alpha<1)$ in some interval, then a_n, $b_n=O(1/n^\alpha)$ $(n\to\infty)$. (iii) If $f(x)$ belongs to a Lipschitz class Lip α $(\frac{1}{2}<\alpha<1)$ in some subinterval, then (*) $\sum (|a_n|+|b_n|)<\infty$. (iv) If, in some interval, $f(x)$ is of bounded variation and belongs to a Lipschitz class Lip α $(0<\alpha<1)$, then (*) holds. These theorems are generalization of Noble's [Math. Ann. **128** (1954), 55–62, 256; MR **16**, 126]. Method of proof is quite different from Noble's. The author uses a theorem essentially due to E. A. C. Paley and N. Wiener [Fourier transforms in the complex domain, Amer. Math. Soc., New York, 1934; inequality (31.10)].

S. Izumi (Sapporo)

4734:

Kennedy, P. B. Fourier series with gaps. II. Quart. J. Math. Oxford Ser. (2) **8** (1957), 84–88.

The author proves the following theorem: Let $0<\eta<\pi$ and $\varphi(t)\uparrow\infty$ as $t\uparrow\infty$. Then there are an increasing sequence (n_k) of integers and a function f in L^2 such that: (1) $\liminf_{k\to\infty} n_k e^{-k(\pi-\eta)/22}\geqq 1$; (2) Fourier coefficients a_n, b_n of f vanish except for $n=n_k$ $(k=1, 2, \cdots)$; (3) $f\in$ Lip α $(-\eta, \eta)$ for every $\alpha<1$; (4) $\limsup (|a_n|+|b_n|)\varphi(n)=\infty$; and (5) $\sum (|a_n|+|b_n|)=\infty$.

This means that in (iii) of part I [reviewed above], the gap condition "$n_{k+1}-n_k\to\infty$" cannot be replaced by "$n_k/k\to\infty$". He remarks that the same holds for (iv) of the above review.

S. Izumi (Sapporo)

4735:

Kennedy, P. B. On the coefficients in certain Fourier series. J. London Math. Soc. 33 (1958), 196–207.

Let E be a set on the interval $(0, 2\pi)$ and $\alpha>0$. A function $f(x)$ is said to belong to Lip $\alpha(E)$ if $f(x+h)-f(x)=O(|h|^\alpha)$ uniformly for x in E as $h\to 0$ through unrestricted real values. A set E is said to have positive spread if there is a number $d>0$ such that, for every $N>1$, E contains N points $x_1, \cdots, x_N$ such that $|x_p-x_q|>d/N$ $(p\neq q)$. Dense sets and sets of positive measure have positive spread.

The author proves the following theorems: (I) If $f(x)$ belongs to Lip $\alpha(E)$, where $0<\alpha<1$ and E is of positive measure; and if its Fourier coefficients a_n, b_n vanish except for $n=n_k$ $(k=1, 2, \cdots)$, where (n_k) satisfies the Hadamard gap condition; then a_n, $b_n=O(1/n^\alpha)$. (II) If $f(x)$ belongs to Lip $\alpha(E)$ $(0<\alpha<1)$, where E has positive spread; and if its Fourier coefficients a_n, b_n vanish except for $n=n_k$, (n_k) satisfying the gap condition

$$(n_{k+1}-n_k)/n_k^\beta \log n_k\to\infty \quad (0<\beta<1);$$

then a_n, $b_n=O(1/n^{\alpha\beta})$ $(n\to\infty)$. The exponent of n in the last expression cannot be replaced by $\alpha\beta+\varepsilon$ for any $\varepsilon>0$. Further, if $\alpha>\frac{1}{2}(\beta^{-1}-1)$, then $\sum (|a_n|+|b_n|)<\infty$. The case $\alpha=\frac{1}{2}(\beta^{-1}-1)$ is open.

S. Izumi (Sapporo)

4736:

Skvorcova, M. G. Some theorems on transformations of Fourier series. Kabardin. Gos. Ped. Inst. Uč. Zap. **12** (1957), 55–61. (Russian)

A sequence $\{\lambda_n\}$ belongs to the class (A, B) if, whenever a Fourier series is in A, using λ_n as multipliers converts it into a Fourier series in B. In this paper V^p denotes the class of functions of finite integrated pth power variation. Let $\Lambda(x)=\frac{1}{2}\lambda_0+\sum_{k=1}^{\infty}\lambda_k\cos kx$ have $(C, 1)$ means $l_n(x)$. The author proves the following theorems. A necessary and sufficient condition for $\{\lambda_n\}\in(L^p, BV)$, $p>1$, is that l_n are uniformly in $V^{p/(p-1)}$, or equivalently $\Lambda(x)\in V^{p/(p-1)}$ if and only if l_n are uniformly in $V^{p/(p-1)}$. If $\{\lambda_n\}\in(L_\phi, C)$ then $\Lambda\in L$; if $\Lambda\in C$, $\{\lambda_n\}\in(L_\phi, C)$; if $\Lambda\in L_\phi$, $\{\lambda_n\}\in(B, L_\phi)$.

R. P. Boas, Jr. (Evanston, Ill.)

4737:

Helson, Henry; et Kahane, Jean-Pierre. Sur les fonctions opérant dans les algèbres de transformées de Fourier de suites ou de fonctions sommables. C. R. Acad. Sci. Paris 247 (1958), 626–628.

For any locally compact abelian group G (for instance, the circle T, the real line R, the integer group Z), let $A(G)$ denote the algebra of the Fourier transforms of the Haar-integrable functions on the dual group of G; for $p=1, 2, 3, \cdots$, $A_R^p(G)$ denotes the set of all functions $f=(f_1, \cdots, f_p)$ on G, with values in the Euclidean space R^p, whose components f_j are real and belong to $A(G)$. A complex function F defined in a set $E\subset R^p$ is said to operate in $A_R^p(G)$ if $F(f)\in A(G)$ whenever $f\in A_R^p(G)$ and the range of f lies in E.

Generalizing a recent result of Katznelson $(G=T, p=1)$ [same C. R. **247** (1958), 404–405; MR **20** #4152], the authors prove theorem 1: If $G=T$, R, or Z, if F is defined in a neighborhood of the origin in R^p, and if F operates in $A_R^p(G)$, then F is real-analytic in some neighborhood of the origin. There are some corollaries, of which the following is perhaps the most interesting: If $G=T$ or R, if F is defined on a compact convex set E in R^p, and if F operates in $A_R^p(G)$, then F can be ex-

tended to a real-analytic function in some open subset of R^p which contains E.

Let $A_R{}^p(n_k)$ be the set of all elements of $A_R{}^p(Z)$ whose support lies in the sequence $\{n_k\}\subset Z$. The proof of theorem 1 also yields theorem 2: If $\{n_k\}$ contains arbitrarily long arithmetic progressions, then theorem 1 holds with $A_R{}^p(n_k)$ in place of $A_R{}^p(Z)$.

The most difficult step in theorem 1 is the proof that the mapping $f\to F(f)$ of $A_R{}^p(G)$ into $A(G)$ carries some sphere about the origin of $A_R{}^p(G)$ into a bounded subset of $A(G)$. *W. Rudin* (New Haven, Conn.)

4738:

Maurin, K. Entwicklung positiv definiter Kerne nach Eigendistributionen. Differenzierbarkeit der Spektralfunktion eines hypoelliptischen Operators. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **6** (1958), 149–155.

Let $S=C_0{}^\infty(\Omega_n)$ be the class of C^∞ functions with compact support on a C^∞ manifold Ω_n, and let (u, v) be an hermitian positive definite bilinear functional on $S\times S$ continuous in $D^m\times D'^m$ topology for some $m=0, 1, \cdots$. If L is a symmetric operator relative to (u, v) and $A\supset L$ is a self-adjoint extension of L, then

$$(u, v)\equiv\int_{\Omega_n\times\Omega_n} K(x, y)u(x)\overline{v(y)}dxdy$$

with $K(x,y)=\int_{\text{spectrum}\,A}\theta(x,y;\lambda)d\mu(\lambda)$ for a suitable $d\mu(\lambda)\geqq 0$. and $K(\cdot,\cdot)\in D'^m(\Omega_n\times\Omega_n)$ Here $\theta(x, y; \lambda)$ is an eigendistribution of L.

This theorem of the author generalizes a previous theorem of Yu. M. Berezanskiĭ [Dokl. Akad. Nauk SSSR **110** (1956), 893–896; MR **19**, 1061] bearing on the case when L is a partial differential operator, and which had been stated as a comprehensive version of Fourier-Stieltjes representation for positive-definite and also completely monotone functions. *S. Bochner* (Princeton, N.J.)

4739:

Mahowald, Mark. A summability theorem in countable toral groups. Math. Ann. **135** (1958), 354–359.

The author proves, for the direct product of infinitely many circle groups, an analog of the classical Abel summability of a Fourier series to its function at a point of continuity. *H. Mirkil* (Hanover, N.H.)

INTEGRAL TRANSFORMS

See also 4754, 4763, 4764, 4901.

4740:

Rooney, P. G. On some theorems of Doetsch. Canad. J. Math. **10** (1958), 421–430.

Let $1<p<\infty$, ω real; the space $H_p(\omega)$ consists of all functions $f(s)$ regular for $\Re s>\omega$ for which

$$\mu_p(f; x)=\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}|f(x+iy)|^p dy\right\}^{1/p}$$

is bounded as $x>\omega$. Let $\lambda>0$; the space $H_{\lambda,p}(\omega)$ consists of all $f(s)\in H_p(\omega')$ for every $\omega'>\omega$ and such that

$$\int_\omega^\infty (x-\omega)^{q\lambda-1}(\mu_p(f; x))^q dx<\infty;\quad 1/p+1/q=1.$$

In the cases $p=1$, $p=\infty$ and $\lambda=0$ certain modifications are needed.

The following principal results are obtained. 1. If $e^{-\omega t}\phi(t)\in L_p(0, \infty)$, $1\leqq p\leqq 2$, $\lambda\geqq 0$, and $f(s)$ is the Laplace transform of $t^\lambda\phi(t)$, $\Re s>\omega$, then $f(s)\in H_{\lambda,q}(\omega)$. 2. If $f\in H_{\lambda,p}(\omega)$, $1\leqq p\leqq 2$, $\lambda\geqq 0$, then there is a ϕ such that $e^{-\omega t}\phi(t)\in L_q(0, \infty)$ and $f(s)$ is the Laplace transform of $t^\lambda\phi(t)$. In the particular case $p=2$, more precise (necessary and sufficient) conditions are achieved. *G. Szegő* (Stanford, Calif.)

4741:

Rooney, P. G. Laplace transforms and generalized Laguerre polynomials. Canad. J. Math. **10** (1958), 177–182.

The author obtains the following necessary and sufficient condition that a function $f(s)$, regular for $\Re s>0$, should be the Laplace transform of a function $t^{\nu/2}F(t)$, $F\in L_2\,(0, \infty)$, $\nu>-1$:

$$\sum_{n=0}^{\infty}\frac{n!}{\Gamma(n+\nu+1)}\left|\sum_{r=0}^{n}\binom{n+\nu}{n-r}f^{(r)}(\tfrac{1}{2})/r!\right|^2<\infty.$$

(The inner sum is, of course, connected with generalized Laguerre polynomials.) If this condition is fulfilled, the latter sum turns out to be

$$=\int_0^\infty |F(t)|^2 dt.$$

An application to Hankel transforms is given. *G. Szegő* (Stanford, Calif.)

4742:

Ivanov, V. V. Some properties of singular integrals of Cauchy type and their applications. Dokl. Akad. Nauk SSSR **121** (1958), 793–794. (Russian)

The integrals considered here are principal values of the form $Sf=(\pi i)^{-1}\int_L (t-t_0)^{-1}f(t)dt$, $t_0\in L$, where L is a simple closed curve. By $Z(p, L)$ the author denotes the class of functions f whose pth derivative is smooth (à la Zygmund) on L; by $\rho_n(f, L)$, the best approximation to f on L by polynomials of degree n. The following theorems are announced. Either of the conditions $Sf\in Z(p, L)$ or $\rho_n(f, L)=O(n^{-p-1})$ is necessary and sufficient for $f\in Z(p, L)$. If $\rho_n(f, L)=O(n^{-p-\alpha})$, then $\rho_n(Sf, L)=O(n^{-p-\alpha})$, $0<\alpha\leqq 1$, and conversely. If f is continuous on L, then $e^{Sf}\in L_p(L)$, $p>0$. The applications are to the Riemann boundary value problem. *R. P. Boas, Jr.* (Evanston, Ill.)

4743:

Malyužinec, G. D. Relationship between Sommerfeld integral inversion formulas and Kontorovič-Lebedev formulas. Dokl. Akad. Nauk SSSR (N.S.) **119** (1958), 49–51. (Russian)

The author considers an integral of the form

$$F(r)=(\pi i)^{-1}\int_\gamma \exp(-ikr\cos\alpha)f(\alpha)d\alpha$$

with $r>0$ and γ a suitable contour in the complex plane. If $F(0)=0$, he shows by formal computation that this can be put into a form considered by the latter named authors in the title. *A. Devinatz* (St. Louis, Mo.)

4744:

Goldberg, Richard R. An inversion of the Stieltjes transform. Pacific J. Math. **8** (1958), 213–217.

The author inverts the transform

$$G(x)=\int_0^\infty (x^2+t^2)^{-1}t\psi(t)dt,$$

and hence the Stieltjes transform, by means of a formula using the Moebius function $\mu(n)$. If $\psi(t)/t\in L(0,\infty)$ and $t^{-1}\psi(t)\log t\in L(0, 1)$, then

$$H(x)=\lim_{N\to\infty} x^{-1}\{\tfrac{1}{2}G(0)+\sum_{k=1}^{N}(-1)^k G(k\pi/x)\}$$

exists for $x>0$, and, for almost all positive t,

$$\varphi(t)=\lim_{p\to\infty}\frac{(-1)^p}{p!}(p/t)^{p+1}\sum_{n=1}^{\infty}\mu(2n-1)(2n-1)^p \times H^{(p)}\{(2n-1)p/t\}.$$

R. P. Boas, Jr. (Evanston, Ill.)

4745:

Ogieveckiĭ, I. I. On the theory of fractional differentiation and integration of periodic functions belonging to an L_p class with $p>1$. Dokl. Akad. Nauk SSSR (N.S.) 118 (1958), 443–446. (Russian)

The author [Ukrain. Mat. Ž. 9 (1957), 205–210; MR 19, 743] and D. Králik [Acta Math. Acad. Sci. Hungar. 7 (1956), 49–64; MR 18, 34] have investigated the Lipschitz classes to which fractional integrals and derivatives belong by using the methods of the constructive theory of functions. Here the author gives an extensive list of similar results for integrated Lipschitz conditions as well as the order of the L^p approximation to a fractional integral or derivative by the partial sums of its Fourier series. *R. P. Boas, Jr.* (Evanston, Ill.)

4746:

Stein, E. M.; and Weiss, Guido. Fractional integrals on n-dimensional Euclidean space. J. Math. Mech. 7 (1958), 503–514.

Let $x=(x_1, x_2, \cdots, x_n)$ denote a general point in Euclidean n-space E^n, $|x|=(x_1^2+\cdots+x_n^2)^{\frac{1}{2}}$, and $dx=dx_1\cdots dx_n$. Let $T_\lambda f\cdot(x)=\int_{E^n} f(y)|x-y|^{-\lambda}dy$. The principal result of the present paper is as follows. Let $0<\lambda<n$, $1<p<\infty$, $\alpha<n-n/p$, $\beta<n/q$, $\alpha+\beta\geqq 0$, $1/q=1/p+[(\lambda+\alpha+\beta)/n]-1$. If $p\leqq q<\infty$, then $\|T_\lambda f\cdot(x)|x|^{\beta}\|_q\leqq A\|f(x)|x|^{\alpha}\|_p$, where A depends upon p, α, β, and λ but not upon f. The demonstration splits into two cases $q=p$ and $q>p$, the former being the more difficult. In the course of the demonstration a number of inequalities are generalized from 1 to n dimensions which are of independent interest. For example, it is proved that if $K(u, v)\geqq 0$ is defined for $u\geqq 0$, $v\geqq 0$, is homogeneous of degree $-n$, and satisfies, for some $p\geqq 1$, $\int_0^\infty K(1, t)t^{n-1-n/p}dt<\infty$, and if $Uf\cdot(x)=\int_{E^n} K(|x|, |y|)f(y)dy$, then $\|Uf\|_p\leqq A\|f\|_p$.

I. I. Hirschman, Jr. (St. Louis, Mo.)

INTEGRAL AND INTEGRODIFFERENTIAL EQUATIONS

See also 4715, 4746, 4782, 4884, 4901, 4922, 4935, 4944.

4747:

Isahanov, R. S. A class of singular integral equations. Soobšč. Akad. Nauk Gruzin. SSR 20 (1958), 9–12. (Russian)

A singular integral equation whose kernel contains the factor $(t-t_0)^{-1}$ is considered. This equation is reduced to a type investigated by I. I. Muskhelishvili [Singular integral equations, OGIZ, Moscow-Leningrad, 1946; MR 8, 586]. Two theorems are given that relate the existence of a solution of the given non-homogeneous equation to the existence of a nontrivial solution of certain homogeneous integral equations. *H. P. Thielman* (Ames, Iowa)

4748:

Gohberg, I. C. On the number of solutions of a homogeneous singular integral equation with continuous coefficients. Dokl. Akad. Nauk SSSR 122 (1958), 327–330. (Russian)

Consider the equation

$$(1)\qquad A\varphi\equiv a(t)\varphi(t)-\frac{b(t)}{\pi i}\int_\Gamma\frac{\varphi(\tau)}{\tau-t}d\tau=0\quad(t\in\Gamma),$$

where Γ is a contour consisting of a finite number of simple, smooth, closed curves with continuous curvature and the solution functions $\varphi(t)$ are considered relative to the space $L_2(\Gamma)$ of complex-valued functions defined on Γ with summable squares. Then it is proved that if $a(t)$, $b(t)$ are continuous and $a^2(t)-b^2(t)\neq 0$ on Γ, equation (1) has exactly $\kappa(A)=(2\pi)^{-1}\int_\Gamma d_t \arg[(a(t)+b(t))/(a(t)-b(t))]$ linearly independent solutions for $\kappa(A)>0$ and only the trivial solution for $\kappa(A)\leqq 0$. This generalizes an earlier statement and proof by S. G. Mihlin [Uspehi Mat. Nauk (N.S.) 3 (1948), no. 3(25), 29–112; MR 10, 305] who had assumed $a(t)$, $b(t)$ to satisfy a Hölder condition.

J. F. Heyda (Cincinnati, Ohio)

4749:

Pokornyĭ, V. V. On the convergence of formal solutions of non-linear integral equations. Dokl. Akad. Nauk SSSR 120 (1958), 711–714. (Russian)

The present article contains a theorem on the nonlinear integral equation of P. S. Uryson

$$\varphi(x)=\int_0^1 K[x, y, \varphi(y); \alpha]dy.$$

This theorem establishes the fact that the formal solutions in the form of an infinite series in terms of fractional powers of the small parameter are actual solutions of the integral equation for small enough values of the parameter. The proof is based on ideas which date back to A. M. Lyapunov and E. Schmidt.

H. P. Thielman (Ames, Iowa)

4750:

Vaĭnberg, M. M. Positive solutions of certain nonlinear integral equations. Moskov. Oblast. Pedagog. Inst. Uč. Zap. 57 (1957), 61–72. (Russian)

The author uses the method of iteration to prove the existence of positive solutions of the nonlinear integral equation

$$F[u(x)]=\int_B K[x, y, u(y)]dy+f(x).$$

Under certain restrictions on the kernel the author shows the existence of continuous solutions and the existence of solutions belonging to $L^{(p)}(B)$, where B is the region of integration. The function $f(x)$ is assumed to be positive. The case $f(x)\equiv 0$ is given special consideration.

H. P. Thielman (Ames, Iowa)

4751:

Nikitin, B. D. Existence of solutions of an infinite system of non-linear integral equations. Moskov. Oblast. Pedagog. Inst. Uč. Zap. 57 (1957), 81–98. (Russian)

The topological method based on Schauder's fixed point theorem is used to prove an existence theorem on the solution of an infinite system of nonlinear integral equations in a Frechet space. The work is part of the author's candidate dissertation prepared under the direction of N. A. Lednev. *H. P. Thielman* (Ames, Iowa)

4752:

Pogorzelski, W. Propriétés d'une intégrale singulière pour les arcs non fermés et son application. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. 6 (1958), 85–87.

A summary of the author's paper reviewed below.

R. C. MacCamy (Pittsburgh, Pa.)

4753:

Pogorzelski, W. Sur l'équation intégrale singulière non linéaire et sur les propriétés d'une intégrale singulière pour les arcs non fermés. J. Math. Mech. **7** (1958), 515–532.

The existence of at least one solution of the non-linear integral equation of the second kind

$$\varphi(t)=\lambda\int_L \frac{F(t,\tau,\varphi(\tau))}{\tau-t}\,d\tau,$$

for sufficiently small λ, is shown under certain conditions on the function F and the path L. The path L consists of a finite number of non-closed simple arcs having no points in common. The arcs have continuous tangents at each point. If $\theta(t_1, t_2)$ denotes the angle between the tangents at two arbitrary points t_1 and t_2 of the arcs L, it is supposed that $\theta(t_1,t_2)<\text{const.}|t_1-t_2|^h$, $0<h\le 1$. The function $F(t,\tau,u)$ is a continuous complex function for $t\in L$, $\tau\in L$, $u\in\Pi$, where Π is the complex plane. F satisfies two conditions: (1) $|F(t,\tau,u)|<m_F|u|^r+m_F'$, (2) $|F(t,\tau,u)-F(t_1,\tau_1,u_1)|<K_F[|t-t_1|^{\mu_1}+|\tau-\tau_1|^{\mu}+|u-u_1|]$, where m_F, m_F', K_F, r, μ_1, μ are constants for which $0<\mu<\mu_1\le 1$; $0<r<1$; $\mu+r<1$; $\mu<h\le 1$.

The proof depends primarily upon the topological fixed point theorem of J. Schauder to the effect that if in a linear Banach space, normed and complete, a continuous transformation makes a correspondence between a convex closed point set E and a compact sub-set of E, then there exists at least one point invariant under the transformation. The chief burden of the paper consists in proving at some length that the conditions of this fixed point theorem are met under the conditions (1) and (2), with a norm chosen as

$$\|\varphi(t)\|=\sup_{t\in L}\Big[\prod_{\nu=1}^{p}|t-a_\nu|^{\alpha+\mu}|t-b_\nu|^{\alpha+\mu}|\varphi(t)|\Big],$$

where $L=\sum_{\nu=1}^{p}a_\nu b_\nu$, $\mu r(1-r)^{-1}<\alpha<1-\mu$, provided λ is sufficiently small.

The author remarks that his method may be extended to the system of singular integral equations

$$\varphi_\nu(t)=\int_L\frac{F_\nu(t,\tau,\varphi_1(\tau),\cdots,\varphi_n(\tau))}{\tau-t}\,d\tau,\ \nu=1,2,\cdots,n.$$

M. S. Robertson (New Brunswick, N.J.)

4754:

Noble, B. Certain dual integral equations. J. Math. Phys. **37** (1958), 128–136.

This paper is concerned with the dual integral equations

$$(1)\qquad \int_0^\infty \xi^{\alpha}\psi(\xi)J_\nu(\xi a)d\xi=f(x)\quad(0<x<1)$$

$$(2)\qquad \int_0^\infty \psi(\xi)J_\nu(\xi x)d\xi=g(x)\quad(x>1).$$

Two cases, $-2<\alpha<0$ and $0<\alpha<2$, are considered.

In case $-2<\alpha<0$, the right hand side of (2) is put equal to an unknown function $\chi(x)$ for $0<x<1$. If (2) is then solved for $\psi(\xi)$ by Hankel's theorem and the result substituted in (1), an integral equation of the form

$$(3)\qquad \int_0^1\lambda\chi(\lambda)K(x,\lambda)d\lambda=F(x)\quad(0<x<1)$$

is obtained, $F(x)$ being a known function and

$$K(x,\lambda)=\int_0^\infty \xi^{1+\alpha}J_\nu(\lambda\xi)J_\nu(x\xi)d\xi.$$

The integral equation (3) is then solved for $\chi(\lambda)$ by using the known solutions of Abel's integral equation. When χ is found, an application of Hankel's theorem to (2) gives ψ.

A similar method is applicable in case $0<\alpha<2$, by interchanging the parts played by (1) and (2). The analysis is formal.

E. T. Copson (Fife)

4755:

Vladimirov, V. S. Equation of transport of particles. Izv. Akad. Nauk SSSR Ser. Mat. **22** (1958), 475–490. (Russian)

The author continuous his study of an integro-differential equation [same Izv. **21** (1957), 3–52, 681–710; MR **20** #2205, #2206] by operator methods in H_p.

M. M. Day (Urbana, Ill.)

FUNCTIONAL ANALYSIS

See also 4580, 4610, 4723, 4740, 5110.

4756:

Monna, A. F. Sur les espaces normés non-archimédiens. III, IV. Nederl. Akad. Wetensch. Proc. Ser. A. **60** = Indag. Math. **19** (1957), 459–476.

For I, II see the author's papers in same Proc. **59** (1956), 475–483, 484–489 [MR **18**, 320]. For further background and definitions, see S. Kasahara, Proc. Japan. Acad. **30** (1954), 572–575 [MR **16**, 832] and I. Fleischer, Nederl. Akad. Wetensch. Proc. Ser. A. **57** (1954), 165–168 [MR **15**, 964] — referred to as F in the sequel. The present papers extend the results of F, showing that if E is a complete nonarchimedean vector space over a complete spherical field K, there exists an orthogonal system $\{y_i\}$, $i\in I$, of (unique) cardinality a such that all $x\in E$ have the unique expansion $x=\sum_{i\in I}a_iy_i$ and $\|x\|=\sup_{i\in I}\|a_iy_i\|$; E is isomorphic with the sequence space $C(a)$ of F. The concept of an orthogonal system $\{y_i\}$ is stronger than mere mutual orthogonality: in addition, it must be true that $y_i\in V_k$ (for all i and k in I, $i\ne k$), where the V_k's are orthogonal supplements of the y_k's. Various consequences of this theorem are considered, as well as conditions for completeness (=linear denseness) and orthogonality. Finally, it is shown that weak and strong convergence of sequences coincide and that E is weakly (sequentially) complete even though the weak and strong topologies are distinct.

G. K. Kalisch (Minneapolis, Minn.)

4757:

Riss, J. Les semi-normes dénombrablement convexes. Publ. Sci. Univ. Alger. Sér. A **3** (1956), 107–120.

Let E be a linear space of real valued functions on a set X and P a semi-norm on E which is countably convex, i.e., $f_n\in E$, $n\ge 0$, $|f_0(x)|\le\sum_{n\ge1}|f_n(x)|$ for every $x\in X$ imply $P(f_0)\le\sum_{n\ge1}P(f_n)$.

The author shows that such a P can be extended to all the functions f such that $|f|\le\sum_n|f_n|$, $f_n\in E$ and $\sum_n P(f_n)<+\infty$; there P is complete and has some properties similar to those of the semi-norm defined by the Lebesgue upper integral.

He also shows that every positive linear functional on E is a sum of a continuous and a singular functional, and the latter is a linear combination of a finite number of the function-values at hyperfilters of X. (A hyperfilter is an ultrafilter which contains every intersection of a countable number of its members.)

I. G. Amemiya (Kingston, Ont.)

4758:

Audin, Maurice. Sur les équations linéaires dans un espace vectoriel. Publ. Sci. Univ. Alger. Sér. A 4 (1957), 5–76.

This is a detailed report of the author's thesis, the results of which were announced previously [C.R. Acad. Sci. Paris **244** (1957), 711–713, 2880–2882; MR **18**, 659; **19**, 663]. *W. A. J. Luxemburg* (Pasadena, Calif.)

4759:

Robertson, Wendy. Completions of topological vector spaces. Proc. London Math. Soc. (3) **8** (1958), 242–257.

In the following, E will denote a real or complex vector space, and ξ and η will denote Hausdorff topologies on E compatible with the vector space structure. Consider ξ, η on E, ξ finer than η, and suppose: (*) every Cauchy filter in ξ which converges in η also converges in ξ. If E is complete under η, (*) of course says it is then complete under ξ. If E is not complete under η, what is the relation between (*) and the completions $\hat{E}_\xi$, $\hat{E}_\eta$ of E under ξ and η respectively? Theorem: (*) is a necessary and sufficient condition for $\hat{E}_\xi \subset \hat{E}_\eta$ (i.e., for the extension $\hat{\imath}$: $\hat{E}_\xi \to \hat{E}_\eta$ of the identity map $i: E \to E$ to be (1-1)). More generally, the author assumes (*) on a subset A of E and studies the relationship between $\hat{A}_\xi$ and $\hat{A}_\eta$, and also between the closure of A under ξ and η respectively.

In the second half of the paper, the author extends the domain of validity of the open-mapping and closed-graph theorems. The common aim in this now active project (cf. V. Pták [Czechoslovak Math. J. 3(78) (1953), 301–364; MR **16**, 262], A. Robertson and W. Robertson [Proc. Glasgow Math. Assoc. **3** (1956), 9–12; MR **18**, 810], H. S. Collins [Trans. Amer. Math. Soc. **79** (1955), 256–280; MR **16**, 1030], and J. L. Kelley [Michigan Math. J. **5** (1958), 235–246]) is to remove the restriction of metrizability, in which case it becomes necessary to find an appropriate generalization of completeness or second category. Unlike the above references, the author is interested in non-locally-convex spaces. She defines E to be ultrabarrelled under η if every ξ with a neighborhood basis around the origin consisting of sets closed in η is coarser than η. If η and ξ in this definition are required to be locally convex, then it reduces to E being barrelled (tonnelé) under η. By partially replacing completeness with ultrabarrelledness, various extensions of the two theorems in question are obtained.

S. Kaplan (Detroit, Mich.)

4760:

Komura, Yukio. On a theorem of A. Grothendieck. Sci. Papers Coll. Gen. Ed. Univ. Tokyo **7** (1957), 169–170.

Let E be a linear space with locally convex topology τ, E' the dual of E, and $\mathfrak{S}$ a system of convex circled bounded sets in E satisfying: (a) if $S_1, S_2 \in \mathfrak{S}$, there exists some $S \supset S_1 \cup S_2$, $S \in \mathfrak{S}$; (b) $\bigcup_{S \in \mathfrak{S}} S = E$. Let $\hat{E}'$ be the totality of linear functionals which are τ continuous on each element S of $\mathfrak{S}$. It is shown that $\hat{E}'$ is the completion of E' with the topology of uniform convergence on each element of $\mathfrak{S}$. In the theorem of Grothendieck [C. R. Acad. Sci. Paris **230** (1950), 605–606; MR **12**, 715], condition (a) was replaced by the condition that all elements of $\mathfrak{S}$ were closed. *R. E. Fullerton* (College Park, Md.)

4761:

Mulholland, H. P.; and Rogers, C. A. Representation theorems for distribution functions. Proc. London Math. Soc. (3) **8** (1958), 177–223.

The purpose of this paper is to give, in certain spaces of distribution functions, an integral representation of an arbitrary distribution function in terms of external distributions. The distribution functions we are interested in form a closed bounded convex set in a certain topological vector space, so we have a sharpening of the Krein-Milman theorem in this case. A somewhat related theory has been given by Choquet [C.R. Acad. Sci. Paris **243** (1956), 699–702; MR **18**, 219].

An example of the type of result proven is: Let $f_1, f_2, \cdots, f_k$ be Borel measurable functions on the real line, and let K be the set of distribution functions satisfying $\int f_r(x)dF(x)=0$ for $r=1, 2, \cdots, k$; then each $F \in K$ is expressible as $F(x)=\int_0^1 H_t(x)dt$, where for each t, H_t is an extreme point of K. The extreme points of K are the step functions having l jumps at $x_1, x_2, \cdots, x_l$, where $1 \leq l \leq k+1$, and where the l vectors $(1, f_1(x_j), \cdots, f_k(x_j))$ are linearly independent. By using this result, the authors can prove the following. Let $g, f_1, f_2, \cdots, f_k$ be Borel measurable functions; let K be as above, and let E be the set of extreme points of K. Then

$$\sup_{F \in K} \int g(x)dF(x) = \sup_{H \in E} \int g(x)dH(x).$$

It is rather difficult to describe the proofs of the theorems in a few lines. However, the methods are essentially constructive and depend on an intricate analysis of the Lebesgue measure theory on the real line. It would be of interest if one could prove some kind of generalization to the case where the real line is replaced by Euclidean space of arbitrary dimension. *L. Ehrenpreis* (Waltham, Mass.)

4762:

Kist, Joseph. Locally o-convex spaces. Duke Math. J. **25** (1958), 569–582.

Let E denote a partially ordered vector space with a locally convex topology $\mathcal{T}$. A convex subset S of E is said to be o-convex if $x \in S$ whenever there exist s and s' in S with $s \leq x \leq s'$. The space E is said to be locally o-convex if there is a fundamental system of neighbourhoods of the origin consisting of o-convex symmetric sets. The author is mainly concerned with homomorphisms (i.e., order preserving linear mappings) of a locally o-convex space, and, in particular, with conditions under which all homomorphisms are continuous. Given a fundamental system $\{V\}$ of convex symmetric neighbourhoods of the origin for the topology $\mathcal{T}$, each V is contained in a smallest o-convex set $N(V)$, and the sets $N(V)$ form a fundamental system for a locally o-convex topology $\mathcal{T}_\eta$. Among some preliminary results it is proved that the dual positive cone P' consists of the zero functional if and only if $\mathcal{T}_\eta$ is the indiscrete topology, and that P' is total in E' if and only if $\mathcal{T}_\eta$ is a Hausdorff topology. In the first main theorem it is proved that if E is Hausdorff and locally o-convex, then every homomorphism of E into a locally o-convex space is continuous if and only if the given topology $\mathcal{T}$ is the finest locally o-convex topology in E; other equivalent conditions are given in terms of duality and o-inductive limits, respectively. The space E is said to be o-bornological if it is locally o-convex and if every bounded homomorphism of E into a locally o-convex space is continuous. Four conditions are given, each of which is equivalent to E being o-bornological. A bounded subset of the positive cone P is called a P-bounded set, and E is said to be P-o-bornological if it is locally o-convex and if every homomorphism of E into a locally o-convex space that takes P-bounded sets into bounded sets is continuous. Four conditions are given, each of which is equivalent

to E being P-o-bornological. In a final theorem it is proved that if E is a vector lattice, and $\mathcal{T}$ is a sequentially complete Hausdorff topology compatible with the lattice structure, then all the conditions mentioned above are equivalent to each other.

F. F. Bonsall (Newcastle-upon-Tyne)

4763:

Olubummo, A. The Laplace-Stieltjes transform of an increasing vector-valued function. Quart. J. Math. Oxford Ser. (2) 8 (1957), 97–107.

Let V be a partially ordered B-space, V^+ the cone of positive elements. It is supposed that V^+ is closed and satisfies a condition of normality, that is, there exists a fixed $\eta>0$ such that $\|u+v\|\geqq\eta\|u\|$ for all u, $v\in V^+$. In such a space a monotone increasing vector-valued function is of bounded variation. The Laplace-Stieltjes integral $\int_0^\infty e^{-\lambda t}dT(t)\equiv f(\lambda)$ is well defined and, if $T(t)$ is monotone, then the abscissa of convergence of the integral is a singularity of the function $f(\lambda)$. This extends the theorem of H. Hamburger [Math. Z. 7 (1920), 302–322] to vector-valued functions. *E. Hille* (New Haven, Conn.)

4764:

Weston, J. D. Functions of bounded variation in topological vector spaces. Quart. J. Math. Oxford Ser. (2) 8 (1957), 108–111.

Let $g(t)$ be a function on reals to a topological vector space X. Let S be a finite succession of non-overlapping closed subintervals $[a_i, b_i]$ of a given interval $[a, b]$, and let

$$v_S(g)=\sum [g(b_i)-g(a_i)].$$

Let $V[g]$ be the set of points $v_S(g)$ for all possible choices of S. Then $g(t)$ is of bounded variation in $[a, b]$ if the set $V[g]$ is bounded. A partially ordered topological space is called an A-space if corresponding to any neighborhood G of the origin there is a neighborhood H of the origin such that $x\in G$ whenever there exist u and v in H with $u\leqq x\leqq v$. If $g(t)$ has values in an A-space and is monotonic in $[a, b]$, then $g(t)$ is of bounded variation in $[a, b]$. If X is a locally convex A-space which is sequentially complete, then a continuous function may be integrated with respect to a monotonic function $g(t)$. The Laplace-Stieltjes integral $f(\lambda)\equiv\int_0^\infty e^{-\lambda t}dg(t)$ may be defined and, if X admits complex scalars, then $f(\lambda)$ is holomorphic in a half-plane. If the positive cone in X is closed, then the abscissa of convergence is a singularity of $f(\lambda)$ when $g(t)$ is monotonic. This generalizes a result of A. Olubummo [see the preceding review] for the case of B-spaces.

E. Hille (New Haven, Conn.)

4765:

Putnam, C. R. On bounded matrices with non-negative elements. Canad. J. Math. 10 (1958), 587–591.

The author considers bounded linear operators A in the real sequential Hilbert space (l^2) such that the corresponding matrix (a_{ik}) has non-negative elements. Let sp(A) denote the spectrum of A, and let μ be the spectral radius $\sup\{|\lambda|: \lambda\in \mathrm{sp}(A)\}$. The following theorems are proved. (I) $\mu\in\mathrm{sp}(A)$. (II) If for some $n\geqq1$ there is a positive diagonal element d of A^n, then $\mu\geqq d^{1/n}$. (III) If $\mu>0$ and is a pole of the resolvent $(A-\lambda I)^{-1}$, then there is at least one characteristic vector $x=\{x_k\}$ belonging to μ with $x_k\geqq0$ $(k=1, 2, \cdots)$. (IV) If for every pair (i, k) there exists an integer $M\geqq1$ such that $(A^M)_{ik}>0$, and if μ is a pole of $(A-\lambda I)^{-1}$, then μ is a simple pole and a simple characteristic number, and there exists a characteristic vector x belonging to μ with $x_k>0$ $(k=1, 2, \cdots)$. As pointed out by the author, (I) has been proved under more general conditions by the reviewer [J. London. Math. Soc. 30 (1955), 144–153; MR 16, 936] and (III) and part of (IV) have been proved by Krein and Rutman [Uspehi Mat. Nauk (N.S.) 3 (1948), no. 1(23), 3–95; Amer. Math. Soc. Transl. no. 26; MR 10, 256; 12, 341] subject to the stronger condition that A be completely continuous.

F. F. Bonsall (Newcastle-upon-Tyne)

4766:

Lippmann, Horst. Metrische Eigenschaften verschiedener Winkelmasse im Minkowski- und Finslerraum. I, II. Nederl. Akad. Wetensch. Proc. Ser. A 61=Indag. Math. 20 (1958), 223–238.

The paper studies the "metrics" for the directions x, y at a point of a Minkowski space induced by various definitions of angles or trigonometric functions found in the literature. For example, $g(x, y)=\frac{1}{2}F[x|F(x)-y|F(y)]$ defines a metrization in the usual sense; $\alpha(x, y)=2$ arc sin $g(x, y)$ satisfies a weakened triangle inequality; the reviewer's sine function sm(x, y) a yet weaker form; arc cos $w(x, y)$, where $w(x, y)$ is Finsler's cosine, leads to a distance which is "fast metrisch" in Menger's sense.

H. Busemann (Cambridge, Mass.)

4767:

Čžan, Čžao-čži. Some remarks on unconditionally convergent series. Advancement in Math. 4 (1958), 560–566. (Chinese. Russian summary)

The following theorem is proved: A Banach space E is finite dimensional if and only if, for functions x defined on $0\leqq t\leqq1$ and taking values in E, the following two properties are equivalent: (1) for every $f\in E^*$, the numerical function $f(x(\cdot))$ is of bounded variation; (2) there exists a finite constant M such that $\sum_i \|x(t_{i+1})-x(t_i)\|<M$ holds for all finite partitions $0=t_0<t_1<\cdots<t_n=1$. The proof makes use of the Dvoretzky-Rogers theorem [Proc. Nat. Acad. Sci. U.S.A. 36 (1950), 192–197; MR 11, 525] that absolute and unconditional convergences in a Banach space are equivalent if and only if the space is finite dimensional. The paper also includes a result concerning unconditional convergence. The second part of the paper is an expository discussion on the most general form of bounded linear operators from space (c_0) into an arbitrary Banach space, or from space (c) into any weakly complete Banach space.

Ky Fan (Notre Dame, Ind.)

4768:

Hirschfeld, R. A. On best approximations in normed vector spaces. Nieuw Arch. Wisk. (3) 6 (1958), 41–51.

Let E denote a normed vector space, G a closed subspace of E, x an element of E. E is said to have property (E) if, for any choice of G and x, the equations $y\in G$, $\|x-y\|=\inf_{g\in G}\|x-g\|$ are soluble. E is said to have property (U) if at most one solution exists. The author reviews known results and adds some new ones. Concerning (E), it is shown that this property is enjoyed by all reflexive Banach spaces (generalizing a result of F. Riesz for Hilbert spaces [Acta Math. 41 (1918), 71–98]); and that any conjugate space E^* has property (E), the subspace G being assumed weakly (sequentially) closed. The proofs are direct and use relative weak (sequential) compactness of bounded sets, coupled with weak lower semicontinuity of the norm.

A necessary and sufficient condition is given in order that E shall have property (U), namely that its norm be strict, i.e., from $\|x+y\|=\|x\|+\|y\|$, $x\neq0$, follows $y=tx$ for some $t\geqq0$. This latter condition is also shown to be equivalent to strict convexity of E, i.e., the demand that

$\|x\|=\|y\|=1$, $x\neq y$ together imply that $\|\lambda x+(1-\lambda)y\|<1$ for $0<\lambda<1$. *R. E. Edwards* (Woking)

4769:

Singer, Ivan. Les points extrémaux de la boule unité du dual d'un produit tensoriel normé inductif d'espaces de Banach. Bull. Sci. Math. (2) **82** (1958), 73–80.

Let E and F be Banach spaces and let $E\check{\otimes}F$ be their inductive normed tensor product, i.e. the completion of $E\odot F$ (the normed tensor product of E and F) under the least crossnorm whose associate is also a crossnorm [R. Schatten, A theory of cross-spaces, Princeton Univ. Press, 1950; MR **12**, 186]. It is shown (theorem 1) that any extreme point of the unit sphere of $(E\check{\otimes}F)^*$ can be represented as $f\otimes g$, where f and g are extreme points of the unit spheres of E^* and F^*, respectively. A partial converse (theorem 2) to this is asserted, the proof of which is based on the following (lemma 3): If A is a compact convex subset of a locally convex space, and if $x\in A$ is not an extreme point of A, then there exist points y and z in A such that y is an extreme point of A and $x=\lambda y+(1-\lambda)z$ for some $\lambda\in\,]0,1[$. {That this is false can be seen by letting A be the unit cell of the space $C^*[0,1]$ (in its weak* topology), letting x be the functional corresponding to Lebesgue measure on $[0,1]$, and recalling the well-known characterization of the extreme points of A.} *R. R. Phelps* (Princeton, N.J.)

4770:

Bucur, Fl. Sur une classe de transformations fonctionelles. Bul. Inst. Politehn. Bucureşti **18** (1956), no. 3–4, 129–131. (Romanian. Russian and French summaries)

Let E and F be two Banach spaces and f a linear transformation from E to F. The following two theorems are proved. 1. If f is completely continuous then it is continuous when E has the weak topology and F the norm topology. 2. If f is continuous in the above sense and E is reflexive then f is completely continuous. Both proofs are straightforward. *S. Foguel* (Berkeley, Calif.)

4771:

Olmsted, John M. H. Completeness and Parseval's equation. Amer. Math. Monthly **65** (1958), 343–345.

Let E be an inner product space. The author shows: if E is separable and is not complete, there exists an orthonormal set which is maximal but with linear closure which is not all of E.

{The reviewer recalls: (i) a proof of the same theorem, but without assuming separability of E, was published earlier in Japanese by Shibuya and Iwabori [Sûgaku, **8** (1956/57), 30–31; MR **20** #1201]; (ii) J. Dixmier [Acta Sci. Math. Szeged **15** (1953), 29–30; MR **15**, 135] has constructed, for each cardinal $\aleph>\aleph_0$, an incomplete E, whose completion has Hilbert-space dimension $\aleph$, such that every maximal orthonormal set in E fails to have linear closure identical with E; (iii) F. J. Wecken constructed an incomplete E whose completion has Hilbert-space dimension c (cardinal of the continuum), but such that every maximal orthonormal set in E is countable [see O. Teichmüller, J. Reine Angew. Math. **174** (1935), 73–124; p. 90].} *I. Halperin* (Kingston, Ont.)

4772:

Moppert, Karl-Felix. Funktionenscharen im L_2. Math. Z. **67** (1957), 474–478.

The functions under consideration are those of the form $f(\mu,t)$ which for $0\leq t\leq 1$ and $N_1<\mu<N_2$ are in L_2 as functions of t. A family of such functions will be called right unconditionally total at N_1 provided that for all sequences $\{\mu_k\}$ on (N_1,N_2) with $\mu_k\to N_1$, the corresponding sequences $\{f(\mu_k,t)\}$ are total in L_2. Left unconditionally total at N_2 and unconditionally total at N are defined analogously (the latter for μ on some full neighborhood of N). Theorem: If, for $k=0,1,\cdots$, $f(\mu,t)$ is k-fold uniformly right L_2-differentiable at $\mu=0$ (in a sense made clear by the author), if the derived functions $(\partial/\partial\mu)^k f(\mu,t)|_{\mu=0}$ are total in L_2, and if the successive Gram determinants formed from these derived functions are non-vanishing, then the family $f(\mu,t)$ is right unconditionally total at $\mu=0$. A partial converse is obtained under rather stringent further restrictions. *M. G. Arsove* (Seattle, Wash.)

4773:

Mrówka, S. On function spaces. Fund. Math. **45** (1958), 273–282.

Various theorems on function spaces are obtained that are similar to, and in some cases generalize, those of Arens [Ann. of Math. (2) **47** (1946), 480–495; MR **8**, 165] and Fox [Bull. Amer. Math. Soc. **51** (1945), 429–432; MR **6**, 278]. Let X, Y and T denote completely regular spaces, and let Y^X be the space of continuous functions from X to Y. In all results except theorem B listed below, Y^X possesses the compact-open topology. The author is concerned for the most part with finding: (1) conditions on X and Y that restrict the cardinality of neighborhood bases of Y^X, or of families of pseudometrics on Y^X that determine its topology; (2) relationships between a weak form of local compactness on X and a type of convergence, associated with joint continuity, of special nets in Y^X; (3) connections between conditions on X, Y and T and the equivalence of the continuity of any function f from $X\times T$ to Y, with the continuity of the function $\mathfrak{f}$ from T to Y^X defined by $[\mathfrak{f}(t)](x)=f(x,t)$. Typical results are: A. If there exists a family of compact subsets of X of power $\leq\mathfrak{m}$, such that every compact subset of X is contained in a set in the family, and the topology of Y is determined by a family of $\mathfrak{m}$ pseudometrics, then the compact-open topology on Y^X is determined by a family of $\mathfrak{m}$ pseudometrics. B. Let Y be the closed unit interval. If there is a topology for Y^X such that the continuity of any f is equivalent to the continuity of $\mathfrak{f}$ (as defined above), for arbitrary T, then X is locally compact. The other main results are similar.

{Half of the proof of theorem V.1 could be replaced by a reference to Fox's lemma 1. There are several misprints.} *C. W. Kohls* (Urbana, Ill.)

4774:

Rudin, Walter. Averages of continuous functions on compact spaces. Duke Math. J. **25** (1958), 197–204.

In Canad. J. Math. **9** (1957), 79–89 [MR **18**, 747], the reviewer had the temerity to conjecture the existence of certain kinds of points in βN (the Stone-Čech compactification of a countably infinite discrete space). The author demolishes this conjecture by proving the following: Let $\{\Omega_i\}$ be a sequence of distinct points in βN; there exists a continuous function f on βN such that the sequence $\{n^{-1}\sum_{i=1}^n f(\Omega_i)\}$ diverges. He observes, in passing, that every infinite compact Abelian group contains an infinite compact metrizable subgroup.

M. Jerison (Princeton, N.J.)

4775:

Chover, J.; and Feldman, J. On positive-definite integral kernels and a related quadratic form. Trans. Amer. Math. Soc. **89** (1958), 92–99.

Let $\rho(t, s)$ be a continuous strictly positive-definite function with $t, s \in J=[a, b]$. Let C be the Banach space of continuous functions f on J with the uniform norm and let B be the dual space of complex Radon measures on J. The relation $R\mu(t)=\int \rho(t, s)d\mu(s)$ defines a continuous operator R from B to C. Let S be the restriction of R to measures of the form $d\mu(s)=h(s)ds$, $h \in H=L_2(J)$. Among other results it is shown that: (i) $\mu(f)=\int f(t)d\mu(t)$ is finite if and only if $f=S^{\frac{1}{2}}g$ for some $g \in H$, and then $\mu(f)=\|g\|^2$; (ii) $R^{\frac{1}{2}}$ is continuous from B to H and $(h, R^{\frac{1}{2}}\mu)=\int S^{\frac{1}{2}}h(s)d\bar{\mu}(s)$ for all $h \in H$, $\mu \in B$; (iii) the range space of R is a proper subset of the range space of $S^{\frac{1}{2}}$.

M. Loève (Berkeley, Calif.)

4776:

Gross, B. Lineare Systeme. Nuovo Cimento (10) **3** (1956), supplemento, 235–296.

This paper deals with the description of the behaviour and the structure of linear systems on a uniform basis. As is well known, there are intimite relations between the different solutions of linear differential equations or systems of linear differential equations which have the form of integral transforms and which can be used to discuss all solutions if one solution is known. This procedure will be especially useful if the differential equation itself is not known. The mathematical formalism is discussed with special reference to electrical circuits, but is of much wider scope. Its most modern applications are, e.g., the dispersion relations of the quantum theory of wave fields, which have been very useful recently in correlating various experimental data as scattering amplitudes, etc. Therefore, a general study of the underlying mathematical methods, which this paper presents, is extremely welcome for applications in many branches of physics. A very convenient and at the same time very general formulation is obtained through the introduction of a distribution function which is the limit of an analytic function. In this way, it is even possible to include δ functions and pseudo functions in a rigorous manner. Many special results are derived for electrical circuits, all of which have counterparts in many fields of physics, especially in the dispersion relations which have been based so far only on the physical causality argument. Now there seems to be a way to put the dispersion relations on a more firm basis. Besides this, applications to relaxation phenomena are considered which today are also widely used.

H. Salecker (Freiburg)

4777:

Ghosh, P. K. On the mathematical foundations of 'physically observable functions'. Bull. Calcutta Math. Soc. **49** (1957), 25–28.

Some time ago Hosemann and Bagchi introduced in physical, non-rigorous terms, the concept of physically observable functions, i.e., of functions defined within a non-vanishing experimental error [Z. Physik **135** (1953), 50–84; MR **14**, 1072]. The present contribution essentially amounts to a more rigorous reformulation of these ideas, using the techniques of the theory of generalized functions of Mikusiński [Fund. Math. **35** (1948), 235–239; MR **10**, 382] and Temple [J. London Math. Soc. **28** (1953), 134; Proc. Roy. Soc. London Ser. A **228** (1955), 175; MR **14**, 989; **16**, 910].

L. Van Hove (Utrecht)

4778:

Ehrenpreis, Leon. Theory of distributions for locally compact spaces. Mem. Amer. Math. Soc., no. 21 (1956), 80 pp.

This Memoir comprises the greater part of the author's Columbia University thesis, 1953. The author takes the point of view that the distribution theory of L. Schwartz is a study of the operators $\partial/\partial x_i$ on Euclidean space and, guided by this idea, he proceeds to generalize distribution theory by replacing Euclidean space with a locally compact Hausdorff space R which is denumerable at infinity (σ-compact) and the operators $\partial/\partial x_i$ with a countable family of local, closed operators that operate on a certain class of functions on R. A surprising amount of distribution theory is carried over to this general setting. Some of the highlights are as follows:

Let R be as above and let $\mathbf{E}^0$ denote the space of continuous, complex valued functions on R equipped with the topology of uniform convergence on the compact subsets of R. D is an "operator on R" if D is a linear or anti-linear mapping whose domain is a subset of $\mathbf{E}^0$ and whose range is a subset of $\mathbf{E}^0$. D is closed if the graph of D is closed in $\mathbf{E}^0 \times \mathbf{E}^0$. Now let $D=(D_i)$ denote a countable family of closed operators D_i on R, such that D contains the identity operator. A space $\mathbf{E}(D)$, corresponding to the Schwartz space $\mathscr{E}$, is constructed as follows. A function f in $\mathbf{E}^0$ "admits" D^k if: (1) For $D_{i_1}, D_{i_2}, \cdots, D_{i_k}$ in D, f is in the domain of $D_{i_1}D_{i_2}\cdots D_{i_k}$; (2) for p any permutation of $1, 2, \cdots, k$ and D_{i_j} as above, $D_{i_1}D_{i_2}\cdots D_{i_k}f=D_{i_{p(1)}}D_{i_{p(2)}}\cdots D_{i_{p(k)}}f$. Let $\{Q_i\}$ be an enumeration of all operators of the form $D_{i_1}D_{i_2}\cdots D_{i_k}$, where Q_0 denotes the identity operator. $\mathbf{E}(D)$ is the space of all functions in $\mathbf{E}^0$ that admit D^k for all k. $\mathbf{E}(D)$ is a vector space and its topology is defined by the countable collection (V_{ji}) of semi-norms, where $V_{ji}(f)=\max\{|(Q_jf)(x)|: x \in K_i\}$ for f in $\mathbf{E}(D)$. (Here (K_i) is a compact exhaustion of R, that is, a countable family of compact subsets of R such that $K_i \subset$ interior K_j $(i<j)$ and $R=\bigcup_{i=1}^{\infty} K_i$). It is proved that $\mathbf{E}(D)$ is a Fréchet space.

In order to introduce the space $\mathbf{D}(D)$, analogous to the Schwartz space $\mathscr{D}$ of testing functions, an additional requirement is imposed on the operators D_i in D. This requirement is that each D_i be a local operator. An operator P on R is a "local operator" if the following conditions are satisfied: If f is any function with compact carrier (support) C in the domain of P, Pf is with compact carrier C'. Further, (1) for any compact set $K \subset R$, there is a compact set $K' \subset R$ (K' independent of f) so that $C \subset K$ implies $C' \subset K'$; (2) for any compact set $K \subset R$ there is a compact set $K' \subset R$ so that $C \cap K' = \phi$ implies $C' \cap K = \phi$. Then if D is a countable family of closed, local operators on R, and K is any compact subset of R, let $\mathbf{D}_K(D)$ denote the set of functions in $\mathbf{E}(D)$ which have their carrier in K. Let $\mathbf{D}_K(D)$ have the topology induced by $\mathbf{E}(D)$; then if $\{K_i\}$ is a compact exhaustion of R, it may be shown that the spaces $\mathbf{D}_{K_i}(D)$ form a sequence of definition for an $(\mathscr{LF})$ space which is denoted by $\mathbf{D}(D)$. The elements of $\mathbf{D}(D)$ are the functions in $\mathbf{E}(D)$ with compact carrier, and it follows from the general theory of $(\mathscr{LF})$ spaces that $\mathbf{D}(D)$ is a complete, locally convex, Hausdorff topological vector space.

The (generalized) distributions are the elements of $\mathbf{D}'(D)$, the space of continuous linear functionals on $\mathbf{D}(D)$. The topology of $\mathbf{D}'(D)$ is the topology of uniform convergence on the compact subsets of $\mathbf{D}(D)$, not the usual strong dual topology of uniform convergence on all bounded subsets of $\mathbf{D}(D)$. {Reviewer's comment: It

should be noted that in the Schwartz case, the topology of bounded convergence and the topology of compact convergence on $\mathcal{D}'$ agree because $\mathcal{D}$ is a Montel space.} The reason for this choice of topology is as follows. If there exists a (Radon) measure p on R such that $f \in \mathbf{D}(D)$ and $\int |f|^2 dp = 0$ implies $f=0$, then there is defined, in a natural way, an injection i of $\mathbf{D}(D)$ into $\mathbf{D}'(D)$; moreover, $i[\mathbf{D}(D)]$ is dense in $\mathbf{D}'(D)$. An example is given to show that $i[\mathbf{D}(D)]$ is not necessarily dense in $\mathbf{D}'(D)$ if $\mathbf{D}'(D)$ has the topology of bounded convergence. The dual, $\mathbf{E}'(D)$, of $\mathbf{E}(D)$ is considered and it is shown that every distribution in $\mathbf{E}'(D)$ has compact carrier. An example is given to show that the converse may not hold, contrary to the situation in the Schwartz case. Structure theorems for distributions are considered. For example, if $S \in \mathbf{E}'(D)$ it is shown that there is a finite sequence of operators Q_{n_i} and a corresponding sequence of (Radon) measures u_i such that $S = \sum Q_{n_i}' u_i$, where Q_{n_i}' denotes the adjoint of Q_{n_i}. A local structure theorem is proved for the distributions in $\mathbf{D}'(D)$.

Four (bilinear) products, involving functions and distributions, are defined. Among them is the so-called direct (tensor) product. The direct product, denoted by $S_1 \otimes S_2$, of two distributions S_1 and S_2 in $\mathbf{E}'(D)$ is defined. It is shown that the direct product is commutative and associative; moreover, $(S_1, S_2) \to S_1 \otimes S_2$ is bilinear and separately (i.e., in each factor) continuous. It is also proven that the map $(S_1, S_2) \to S_1 \otimes S_2$ is hypocontinuous (in the sense of Bourbaki), but the question of continuity (in both factors) is left open. {Reviewer's comment: $\mathbf{E}'(D)$ is equipped with the topology of compact convergence, but if it happens that this topology agrees with the topology of bounded convergence, then it follows immediately from the work of Grothendieck on (DF) spaces that $(S_1, S_2) \to S_1 \otimes S_2$ is continuous. [See Summa Brasil. Math. 3 (1954), 57–123; MR 17, 765.]}

Generalized translation, convolution, and contraction mappings are defined so that they are brought under the framework of distribution theory. A method is outlined whereby one could construct a space serving the same purpose as the space $\mathcal{S}$ in the Schwartz theory; the generalized Fourier transform is described on this space. It is stated that some of the important results of Fourier analysis can be carried over to this general setting. For example, Bochner's theorem on positive definite functions and the Paley-Wiener theorem.

{Reviewer's comments: Although there are a few examples given, it would be desirable to have some more significant examples worked out to illustrate the general theory. The author's use of the word "semi-reflexive" (e.g., on p. 29) does not agree with another usage often found in the literature.} *R. Dowds* (Lafayette, Ind.)

4779:

Leptin, Horst. Note zur lokalen Struktur der Distributionen. Math. Ann. **135** (1958), 360–368.

En analysant un théorème de L. Schwartz selon lequel toute distribution à support compact est une somme finie de dérivées de mesures, l'auteur établit une généralisation de ce théorème, où $\mathcal{D}$, les dérivations D^p, la topologie de la convergence uniforme et la famille des espaces $\mathcal{D}_K$ sont remplacés resp. par une espace vectorielle arbitraire E, un système d'endomorphismes dans E, une topologie localement convexe sur E et une famille filtrante de sous-espaces de E. A cet effet il étudie certaines applications qui font correspondre à toute topologie sur E une autre (sur E); par exemple, pour un système M d'endomorphismes dans E on définit l'application $M^\cdot$ qui à toute topologie $\mathcal{T}$ sur E fait correspondre la moins fine topologie $\mathcal{T}M^\cdot$ (sur E) de celles qui rendent continus tous les endomorphismes $q \in M$ de E dans $E(\mathcal{T})$; une formule disant que $E(\mathcal{T}M^\cdot)'$ coïncide avec l'espace vectoriel engendré par les éléments ${}^t qy$, où $y \in E(\mathcal{T})'$ et ${}^t q$ est la transposée d'un $q \in M$, jouit un rôle central. En prenant pour E un espace de fonctions continues à support compact sur un espace topologique localement compact X, il obtient un théorème correspondant pour les distributions sur X [voir l'oeuvre d'Ehrenpreis analysée ci-dessus].

S. Łojasiewicz (Cracow)

4780:

Mazur, S.; and Orlicz, W. On some classes of linear spaces. Studia Math. **17** (1958), 97–119.

In 1932, W. Orlicz introduced new classes of functions and sequences, generalizing the well-known l^p, L^p-spaces as follows: Let $M(u)$ be a non-negative even continuous convex function of u defined for all real values and vanishing at $u=0$. He considered the class l^M of all sequences $x=\{t_\nu\}$ for which $\bar{p}(x)=\sum_{\nu=1}^{\infty} M(|t_\nu|)$ is finite and the class L^M of measurable functions $x(t)$ on the interval (a, b) for which $p(x)=\int_a^b M(|x(t)|)dt$ is finite. Then l^M $[L^M]$ is a linear space, and, as it turns out, a Banach-space, in which norm convergence is equivalent to p-convergence (i.e., $x_n \to x$ if $p(x_n - x) \to 0$ as $n \to \infty$) if and only if M satisfies the condition $\delta_2 : M(2u) \leq kM(u)$ for sufficiently small u, with k a constant $[\Delta_2 : M(2u) \leq KM(u)$ for sufficiently large u, with K a constant]. In this paper the authors consider an arbitrary non-negative function $N(u)$ defined for all real values and the classes X^N of all sequences $x=\{t_\nu\}$ [of all measurable functions $x(t)$ on the interval (a, b)] for which $p(x)=\sum_{\nu=1}^{\infty} N(t_\nu)$ converges [for which $p(x)=\int_a^b N(x(t))dt$ exists and is finite]. For these classes the following problems are considered and solved. (a) When are the spaces X^N linear? (b) Which are the necessary and sufficient conditions for the function N in order to make it possible to define a F-norm or a B-norm in the linear space X^N such that the relation $\|x_n\| \to 0$ is equivalent to $p(x_n) \to 0$? Since the solutions of these problems are too long to reproduce here, we will content ourselves with quoting the following theorem, which is a consequence of their work. If N satisfies the following two conditions: (*) $\lim N(t_n)=0$ if and only if $\lim t_n = 0$, (O) $\lim N(t_n) = +\infty$ if and only if $\lim |t_n| = +\infty$; then X^N is a Banach space in which norm convergence is equivalent to p-convergence if and only if there exists a continuous even convex function $M(u)$ vanishing only at $u=0$ and satisfying Δ_2 in the case of function spaces and δ_2 in the case of sequence-spaces, such that $L^M = X^N$.

W. A. J. Luxemburg (Pasadena, Calif.)

4781:

Toscano, Letterio. Relazioni su gli operatori del tipo

$$x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + \cdots + x_m \frac{\partial}{\partial x_m}.$$

Bul. Inst. Politehn. Iaşi **4** (1949), 196–202.

The author obtains formally a large number of relations involving the operators $T_1 = \sum_{i=1}^{m} X_i A_i$, $T_2 = \sum_{i=1}^{m} A_i X_i$ where $\{A_i\}$, $\{X_i\}$ are linear operators over an infinite dimensional linear space and the $\{A_i\}$ and the $\{X_i\}$ commute among themselves, $A_i X_j = X_j A_i$ for $i \neq j$ and $A_i X_i - X_i A_i = I$ for all i. These formulas are applied in particular to the case in which the space is a space of differentiable functions of m variables $(x_1, x_2, \cdots, x_m)$ and $A_i = \partial/\partial x_i$, $X_i = x_i I$, and various formulas in factorial analysis are derived by operating on particular functions.

{There are several confusing typographical errors.}

R. E. Fullerton (College Park, Md.)

4782:

Widom, Harold. On the eigenvalues of certain Hermitian operators. Trans. Amer. Math. Soc. **88** (1958), 491–522.

The first part of the paper concerns the eigenvalues $\lambda_{1n} \geqq \lambda_{2n} \geqq \cdots \geqq \lambda_{n+1,n}$ of the $(n+1)$st section $T_n = (c_{j-k})$, where $j, k = 0, \cdots, n$, of the real, symmetric, infinite Toeplitz matrix (c_{j-k}), where $j, k = 0, 1, \cdots$. Let $f(\theta) \sim \sum c_j e^{ij\theta}$ be real, continuous, even; $\max f(\theta) = M = f(0)$ and $f(\theta) = M$ implies $\theta = 0 \pmod{2\pi}$; $f(\theta)$ has continuous derivatives up to the fourth order near $\theta = 0$ and $\sigma^2 = -f''(0) > 0$. Then, for a fixed k, $\lambda_{kn} = M - \frac{1}{2}[\sigma\pi k/(n+1)]^2[1+\alpha/(n+1)] + o(n^{-3})$ as $n \to \infty$, where $2\pi\alpha = \int_{-\pi}^{\pi} \csc^2 \frac{1}{2}\theta \log[\frac{1}{2}\sigma^{-2}(M-f(\theta))\cot^2 \frac{1}{2}\theta]d\theta$. The proof is first given for the case that $f(\theta) = F(e^{i\theta}) = c_{-k}e^{-ik\theta} + \cdots + c_k e^{ik\theta}$ is a trigonometric polynomial such that the non-real zeros of $f(\theta) - M$ are simple. Let $P(z) \equiv P(z, \lambda) = z^k(F(z) - \lambda)$. It is shown that the eigenvalues of T_n near to M are the zeros, near M, of a certain expression D_1 given explicitly in terms of the zeros of $P(z)$ and the values of $P'(z)$ at these points. Asymptotic formulae for the zeros of D_1 are obtained, via Rouché's theorem, by careful estimates. The case of a general $f(\theta)$ is then obtained from the first case by a suitable approximation.

The second part of the paper concerns the extreme eigenvalues $\lambda_{1A} \geqq \lambda_{2A} \geqq \cdots$ of the integral equation $\int_{-A}^{A} \rho(x-y)\varphi(y)dy = \lambda\varphi(x)$. Let $\rho(x) \in L^1(-\infty, \infty)$ be such that $F(y) = \int_{-\infty}^{+\infty} e^{ixy}\rho(x)dx$ is real, even, $O(|y|^{-3})$ at ∞; $\max F(y) = M = F(0)$ and $F(y) < M$ for $y \neq 0$; $F(y)$ has continuous first and second derivatives near $y = 0$ and $\sigma^2 = -F''(0) > 0$. Then, for a fixed k, $\lambda_{kA} = M - \frac{1}{2}(\sigma\pi k/2A)^2 + o(A^{-2})$ as $A \to \infty$. If, in addition, $F(y)$ has continuous third and fourth order derivatives near $y = 0$, then $\lambda_{kA} = M - \frac{1}{2}(\sigma\pi k/2A)^2(1+\alpha/A) + o(A^{-3})$ as $A \to \infty$, where $\alpha = \alpha(F)$ is constant given by an integral involving F. The proof parallels that of the first part. The main case is that of a rational function $F(y)$ which has simple poles and is such that $F(y) - M$ has only simple zeros, except for $y = 0$.

P. Hartman (Baltimore, Md.)

4783:

Haimovici, Adolf. Sur certains problèmes aux limites résolvables par la méthode de la séparation des variables. An. Şti. Univ. "Al. I. Cuza" Iaşi. Secţ. I (N.S.) **3** (1957), 45–51. (Russian and Romanian summaries)

The author applies the method of the separation of variables to linear problems of the type $Au = A_1u - A_2u = 0$. Two cases are treated. 1) A_1 is a completely continuous self-adjoint linear operator which is applicable to a space of continuous functions $v(x)$, $x \in \Omega_1 \subset R^{n_1}$. A_2 is an analytic differential operator of the Cauchy-Kowalevsky type operating on analytic functions $w(y, t)$, $y \in \Omega_2 \subset R^{n_2}$, $t \in I \subset R$. 2) A_1 and A_2 are completely continuous self adjoint transformations operating on continuous functions defined in Ω_1 and Ω_2, respectively. Solutions are obtained in infinite series of products as usual. In case 1), if $\lambda = 0$ is not a characteristic value of A_1, a unique solution $u(x, y, t)$ is found with standard boundary conditions. The non-homogeneous equation is also solved. In case 2), the equation has a solution only if the spectra of A_1 and A_2 have non-void intersection. The solutions are then obtained from the manifold generated by the functions $v(x) \cdot w(y)$, where v and w satisfy $A_1v = \lambda v$, $A_2w = \lambda w$.

E. R. Lorch (New York, N.Y.)

4784:

Krasnosel'skiĭ, M. A.; and Kreĭn, S. G. Continuity conditions for a linear operator in terms of properties of its square. Voronež. Gos. Univ. Trudy Sem. Funkcional. Anal. **1957**, no. 5, 98–101. (Russian)

In this paper the authors consider a triple of Banach spaces E, H, E^*, where each is a dense linear subset of the next, where H is a Hilbert space, where E^* is the conjugate space of E, and where for each x in E, y in H, the inner product (x, y) agrees with the value $y(x)$ of the linear functional y at the point x. If A is a continuous linear operator in H and if A^* is its conjugate, let $B = AA^*$. Theorem 1 [2]: Let B have a [completely] continuous linear extension defined from E into E^*; then A is defined from H into E^* and is [completely] continuous. Application is made to integral equations in Orlicz spaces.

M. M. Day (Urbana, Ill.)

4785:

Hirschfeld, Rudi. Sur les semi-groupes de transformations de Reynolds. C. R. Acad. Sci. Paris **245** (1957), 1493–1495.

L'Auteur considère un semi-groupe abélien G de transformations de Reynolds T [Kampé de Fériet, Proc. Internat. Cong. Mathematicians, Amsterdam, 1954, vol. III, pp. 237–242, Noordhoff, Groningen, North Holland, Amsterdam, 1956; MR **19**, 490] dans un anneau de fonctions A. Il suppose qu'il est borné, c'est-à-dire, qu'il existe une constante $C \geqq 1$ telle que

$$|Tf(x)| < c \sup_{x \in X} |f(x)|, \; T \in G, \; f \in A.$$

A l'aide d'un théorème ergodique, il démontre le théorème suivant: Soit A un espace vectoriel de fonctions f, bornées en mesure dans un espace de mesure, et complet pour la topologie de la convergence simple. Supposons que l'anneau contienne 1. Soit $\bar{A}$ l'ensemble des f égales à leur moyenne $\bar{f}$, A_0 l'ensemble des g de moyenne nulle. Alors: 1) $\bar{A}$ est un sous-anneau de A; 2) si $f \in \bar{A}$, $g \in A_0$, $fg \in A_0$; 3) f admet la décomposition unique $f = \bar{f} + f_0$, $\bar{f} \in \bar{A}$, $f_0 \in A_0$, avec $A = \bar{A} + A_0$.

J. Bass (Paris)

4786:

Rickart, C. E. An elementary proof of a fundamental theorem in the theory of Banach algebras. Michigan Math. J. **5** (1958), 75–78.

The author derives Gel'fand's formula for the spectral radius without appealing to analytic function theory. Instead, he uses elementary properties of roots of unity.

M. Jerison (Princeton, N.J.)

4787:

Waelbroeck, L. Note sur les algèbres du calcul symbolique. J. Math. Pures Appl. (9) **37** (1958), 41–44.

In an earlier paper [same J. (9) 33 (1954), 147–186; MR 17, 513] the author developed a theory of analytic functions in topological algebras. The underlying assumption was the completeness of the (locally convex) algebra and the two-variable continuity of the product ab. In this note, the author shows that weaker condition,s quasi-completeness (completeness of bounded closed sets) and continuity of the product ab in each variable, suffice. The basic observation is this: If B_1 and B_2 are bounded sets, then B_1B_2 is bounded. With this, convergence arguments in integration, etc., can be validated.

B. R. Gelbaum (Minneapolis, Minn.)

4788:

Green, H. F. Rings of infinite matrices. Quart. J. Math. Oxford Ser. (2) **9** (1958), 73.

Let $\Sigma(\alpha)$ denote the set of all infinite matrices which map a sequence space α into itself, the series which arise in the transformations being absolutely convergent. If ϕ denotes the space of all finite sequences, and C that of all stationary sequences, it is known that $\Sigma(\alpha)$ is a ring when α contains ϕ and is normal, whereas $\Sigma(C)$ is not closed under multiplication and is therefore not a ring. The object of this note is to prove the following result. If α contains ϕ and $\Sigma(\alpha)$ is closed under multiplication, then $\Sigma(\alpha)$ is a ring. *R. G. Cooke* (London)

4789:

Mackey, George W. Unitary representations of group extensions. I. Acta Math. **99** (1958), 265–311.

His work on induced representations of locally compact groups has led the author to the study of unitary ray representations. An equally important reason why ray representations should be studied comes from theoretical physics: If the final form of quantum electrodynamics is ever going to have anything to do with Hilbert spaces and group invariance, it will most probably involve ray representations rather than ordinary representations. This is certainly the case of present day quantum mechanics, so that it is all the more surprising how little attention has been paid to ray representations of even such groups as the Lorentz group. Since the article of I. Schur [J. Reine Angew. Math. **132** (1907), 85–181] which reduced the problem of ray representations of finite groups to that of ordinary representations of certain group extensions, very little attention seems to have been paid to this subject until a recent paper by V. Bargmann [Ann. of Math. (2) **59** (1954), 1–46; MR **15**, 397] and Wigner's treatment of the Lorentz group [ibid. **40** (1939), 149–204]. In the present paper the subject is treated in its full generality for the first time. Let G be an arbitrary separable locally compact group. By a projective (or ray) representation L of G is meant a mapping $x \rightarrow L(x)$ of G into the group of unitary operators of some Hilbert space such that: (a) $L(e)=I$; (b) $L(xy)=\sigma(x, y)L(x)L(y)$, where $\sigma(x, y)$ is a complex-valued function on $G\times G$ uniquely determined by L; and (c) the inner product $(L(x)\varphi, \psi)$ is a Borel function of x. For a fixed choice of the multiplier σ the theory is quite analogous to that of ordinary unitary representations ($\sigma(x, y)\equiv 1$). The general theory, largely reducing the study of arbitrary representations to that of irreducible and factor representations, extends without essential change to projective representations. In particular, von Neumann's theorem on the decomposability into factors and the reviewer's theorem on the decomposability into irreducible representations carry over. This follows from the following theorem (which in the case of finite groups goes back to I. Schur, loc. cit.). Given any multiplier $\sigma(x, y)$, the author constructs an extension G^σ of G by the circle group and shows that there exists an ordinary representation L^0 of G^σ, corresponding to any σ-representation L of G, such that the correspondence $L \rightarrow L^0$ is one-one. Using this result, a decomposition theory of σ-representations is obtained. This, in turn, enables the author to establish a theory of induced σ-representations, including systems of imprimitivity.

It is remarkable that when σ_1 and σ_2 are distinct multipliers for G, the theory of the σ_1-representations of G can be as different from the theory of the σ_2-representations of G as the ordinary representation theories of two different groups. For example, the author shows that there exist commutative groups G which have factor representations which are not of type I. There also exists a commutative group G and a multiplier σ such that G has only one irreducible σ-representation which is even infinite-dimensional. On the other hand, if there exists a Borel function $\rho(x)$ from G to the complex numbers of absolute value one such that two multipliers σ_1 and σ_2 are related by $\sigma_2(x, y)=\sigma_1(x, y)\rho(xy)/\rho(x)\rho(y)$, then there is a complete parallelism between the σ_1-representations and the σ_2-representations of G. Indeed if L is a σ_1-representation of G, then $L'(x)=\rho(x)L(x)$ is a σ_2-representation. Moreover, $L\times L'$ is a one-one correspondence between the σ_1-representations and the σ_2-representations of G, which preserves equivalence, irreducibility, etc. This leads to the definition of similarity of multipliers. It is clear that the multipliers form a commutative group and the similarity classes a factor group. {The author does not mention the relationship of this with the cohomology theory of groups.} When G is a direct product the author shows hows the multipliers of G can be obtained from those of the direct factors and certain homomorphisms. But it is shown by an example that the (projective) σ-representations of a direct product need not be related in a simple way to the (projective) σ-representations of the factors, even when the factors have type I σ duals. Thus the foundations have been laid for an extensive theory of projective representations. It is to be hoped that this will lead to a recovery of this subject from its undeserved oblivion. *F. I. Mautner* (Paris)

4790:

Berberian, S. K. $N\times N$ matrices over an AW^*-algebra. Amer. J. Math. **80** (1958), 37–44.

It is shown that if A is an AW^*-algebra, then the algebra A_n of all $n\times n$ matrices, with entries from A, is also an AW^*-algebra. This has been known previously under the restriction either that A is of type I or that A is purely infinite and continuous. (A continuous AW^*-algebra is one containing no abelian projections.) The remaining case is a finite algebra of type II, and it is this case which is settled here. This means that the theorem is true without restriction as to type of A. If A has type, then A_n has the same type.

F. B. Wright (New Orleans, La.)

CALCULUS OF VARIATIONS

See also 4934.

4791:

Cinquini, Silvio. Sopra i fondamenti di una classe di problemi variazionali dello spazio. Rend. Circ. Mat. Palermo (2) **6** (1957), 271–288.

It is well known that in order that the curvilinear integral in the plane

$$\int_C \Phi[x(t), y(t), x'(t), y'(t), x''(t), y''(t)]dt,$$

satisfying the usual positive homogeneity condition in the variables x', y', shall be invariant under change of parameter t, it is necessary and sufficient that $\Phi=F(x, y, x', y', \theta')$, where θ' is curvature. This result is generalized to curvilinear integrals in 3-space. The analogous question when Φ also involves third order derivatives x''', y''', z''' is also settled. The role of θ'

is taken by kB, where k is curvature and B the binormal. Thid order derivatives enter in the form $d(kB)/ds$.

W. H. Fleming (Providence, R.I.)

4792:

De Giorgi, Ennio. Sulla proprietà isoperimetrica dell'ipersfera, nella classe degli insiemi aventi frontiera orientata di misura finita. Atti Accad. Naz. Lincei. Mem. Cl. Sci. Fis. Mat. Nat. Sez. I (8) **5** (1958), 33–44.

The isoperimetric property of the n-ball is established in the class of sets of n-space which have (an oriented) finite perimeter in a sense previously defined by the author [Ricerche Mat. **4** (1955), 95–113; MR **17**, 596]. It should perhaps be pointed out that the restriction to sets is not entirely natural, when oriented boundaries are introduced, and that bounded sets of finite perimeter are, in fact, included in the reviewer's generalized solids, the n-analogues of generalized surfaces, for which the theorem is a simple application of partial area and symmetrization with respect to an axis [US Army Math. Res. Center, Univ. of Wisconsin Rep. **27–30**, June, 1958]. However, the author's result is not included, since he does not assume boundedness, and his proof is of independent interest, since he uses the (less obvious) symmetrization with respect to a hyperplane.

L. C. Young (Paris)

GEOMETRIES, EUCLIDEAN AND OTHER

See also 4928.

4793:

Burde, Gerhard. Zur Einführung der Streckenrechnung. Math. Ann. **135** (1958), 279–282.

Hilbert's algebra of segments [D. Hilbert, Grundlagen der Geometrie, 8th ed., Teubner, Stuttgart, 1956; MR **18**, 227; pp. 60–68] is here based on the concurrency of the altitudes in a triangle instead of on the Pappus-Pascal theorem. For this purpose, Hilbert's definition of the product c of two segments a, b is replaced by the following: Let Ox, Oy be perpendicular axes, E'' a unit-point on the negative y-axis, A, B, points on Ox such that $OA=a$, $OB=b$; then $c=OC$, where C is the intersection of Oy with the perpendicular through B on AE''.

F. A. Behrend (Victoria)

4794:

Gheorghiţă, Şt. I. Généralisation de quelques problèmes d'algèbre vectorielle. Gaz. Mat. Fiz. Ser. A (N.S.) **10**(63) (1958), 454–462. (Romanian. French and Russian summaries)

"Utilisant le calcul vectoriel, on résout le problème suivant: Soit un polygone convexe quelconque à n côtés avec les sommets en A_j $(j=1, 2, \cdots, n)$. Sur chaque côté on prend un point B_j qui divise celle-ci dans le rapport s_j. Désignant par C_j l'intersection des droites A_jB_{j+1} et $A_{j+1}B_{j+2}$, trouver la surface du polygone convexe avec les sommets en C_j $(j=1, 2, \cdots, n)$."

Du résumé de l'auteur

4795:

Szász, Pál. Über den Satz von Wolfgang Bolyai über die Zerschneidung geradliniger ebener Vielecke. Mat. Lapok **7** (1956), 230–237. (Hungarian. Russian and German summaries)

"Der Satz von Wolfgang Bolyai [siehe T. Varga, Mat. Lapok **5** (1954), 101–114; MR **17**, 72], laut welchem ein geradliniges einfaches Vieleck in der Ebene mit einem Rechteck von gegebener Grundlinie zerlegungsgleich ist, wird in vorliegender Arbeit von neuem bewiesen."

Aus der Zusammenfassung des Autors

4796:

Ionescu-Ţiu, C. Contribution à l'étude de quelques inégalités. Gaz. Mat. Fiz. Ser. A (N.S.) **10** (63) (1958), 358–365. (Romanian. French and Russian summaries)

"L'auteur établit que l'aire et le périmètre du triangle $A_1B_1C_1$, obtenu en unissant les points d'intersection des bisectrices d'un triangle quelconque ABC avec le cercle circonscrit, sont plus grands que l'aire et le périmètre du triangle ABC. Répétant cette construction on obtient une suite de triangles $A_nB_nC_n$ dont la limite est un triangle équilatéral. Utilisant ces résultats, on déduit de nombreuses inégalités trigonométriques et algébriques, correlatives aux relations géométriques entre les différents éléments d'un triangle."

Résumé de l'auteur

4797:

Molnár, Ferenc. Sur les points principaux du tetraèdre. I, II. Köz. Mat. Lapok **16** (1958), 1–6, 33–38. (Hungarian)

4798:

Deaux, R. Sur des cubiques planes. Nieuw Arch. Wisk. (3) **5** (1957), 63–67.

In an earlier issue of the same periodical [**4** (1956), 132–139; MR **18**, 755], J. H. Tummers proved the following theorem: If $P_1P_2P_3$, $Q_1Q_2Q_3$ are the cevian triangles of the given points P, Q, for a triangle ABC, the six lines joining the points P_1, P_2, P_3, Q_1, Q_2, Q_3, respectively, to the points (P_2P_3, AQ), (P_3P_1, BQ), (P_1P_2, CQ), (Q_2Q_3, AP), (Q_3Q_1, BP), (Q_1Q_2, CP) meet in a point R.

The point R should be credited to G. Franke, according to Deaux, the latter having given a synthetic proof of Franke's theorem in 1929 and having pointed out, among other things, that the point R is the pole of the line PQ for the conic determined by the points A, B, C, P, Q.

In connection with the theorem quoted, Tummers considered certain cubics which remain invariant under the isogonal transformation relative to the triangle ABC. Deaux considers the analogous cubics which remain invariant under a quadratic transformation having for double points the vertices of a complete quadrangle having ABC for diagonal triangle.

N. A. Court (Norman, Okla.)

4799:

Bottema, O. Une construction par rapport à un triangle. Nieuw Arch. Wisk. (3) **5** (1957), 68–70.

The author proves analytically that the point R is the pole of the line PQ for the conic $ABCPQ$ [see the preceding review].

N. A. Court (Norman, Okla.)

4800:

Goormaghtigh, R. Extension au polygone inscriptible de la droite de Simson d'angle quelconque. Mathesis **67** (1958), 23–27.

The orthogonal projections of a point M of a circle upon the four Simson lines of M for the four triangles determined by the vertices of an inscribed quadrangle taken three at a time lie on a straight line, the Simson line of M for the quadrangle. The orthogonal projections of M upon the Simson lines of M for the five quadrangles determined by the vertices of an inscribed pentagon lie on a straight line, the Simson line of M for the pentagon, etc.

Using complex coordinates the author shows that the propositions remain valid for oblique projections of any angle. Moreover, the angle of projection does not have to

be the same in the successive projections, nor will the final Simson line of M change if the order in which the different angles are used is changed.

N. A. Court (Norman, Okla.)

4801:

Blum, Richard. On a pointwise construction of the lemniscate. Canad. Math. Bull. 1 (1958), 1–4.

This paper deals with a pointwise construction of the lemniscate. If N and $\mathcal{T}$ are two circles of equal radius, externally tangent to each other and having their centers at O and F, respectively, let η be their common tangent with common point of contact and ξ a variable tangent to $\mathcal{T}$, intersecting η and N in Ω and B, respectively. The point of intersection of the straight line through O parallel to ΩF with the perpendicular through B to ξ is a point V of the lemniscate which has its center at O and one of its foci at F.

F. Şemin (Istanbul)

4802:

Strazzeri, Vittorio. L'iperbole equilatera e la determinazione di $\frac{1}{3}$, $\frac{1}{5}$ e $\frac{1}{7}$ di un angolo. Atti Accad. Sci. Lett. Arti Palermo. Parte I (4) **16** (1955/56), 157–162 (1957).

With the help of ruler, compass and an equilateral hyperbola $\alpha/3$ is constructed for a given angle $\alpha<\pi/2$, as in Pappus. A similar construction gives $\alpha/5$.

S. R. Struik (Cambridge, Mass.)

4803:

Fox, Charles. The Pascal line and its generalizations. Amer. Math. Monthly **65** (1958), 185–190.

The author first proves a converse of Pascal's theorem: Given a conic C and a coplanar line l it is always possible to inscribe a hexagon in C whose Pascal line is l. This can be established with four constants to spare. Then the theorem is extended as follows: Given a plane cubic curve C and a coplanar line l one can describe an octogon consisting of four lines a_m and four lines b_n, such that the points (a_m, b_n) are on C if $m\neq n$ and on l if $m=n$. The proof is given by counting equations and unknowns, and the author has still three constants at his disposal. Analogous theorem for the quartic curve; there is one free constant in this case. Impossibility of extension beyond the quartic. The paper ends with a proof of the converse of Salmon's generalization of Pascal's theorem to three dimensions.

O. Bottema (Delft)

4804:

McMahon, James J. Matrix proof of Pascal's theorem. Amer. Math. Monthly **65** (1958), 24–27.

The proof enables the author to give a generalization of the theorem to space of three dimensions.

O. Bottema (Delft)

4805:

Gans, David. Models of projective and Euclidean space. Amer. Math. Monthly **65** (1958), 749–756.

"These three-dimensional models are generalizations of two-dimensional ones, possessing similar advantages, which have already been described in this Monthly [**61** (1954), 26–30; MR **15**, 460]." *From the introduction*

4806:

Crampe, Sibylla. Angeordnete projektive Ebenen. Math. Z. **69** (1958), 435–462.

This paper develops a theory of order for projective planes without the assumption of any incidence theorems. The properties are expressed in terms of a coordinatizing simply ordered set whose elements form a ternary ring with ternary operation designated as $T(u, x, v)$. The author requires the following properties:

(1) $T(u, x, v_1)<T(u, x, v_2)$ for $v_1<v_2$

(2) $T(u_1, x, v_1)<T(u_2, x, v_2)$ for $u_1<u_2$, $x>x_s$
$T(u_1, x, v_1)>T(u_2, x, v_2)$ for $u_1<u_2$, $x<x_s$,

where x_s is determined by $T(u_1, x_s, v_1)=T(u_2, x_s, v_2)$. When the minor theorem of Desargues with axis the line at infinity holds, then $T(u, x, v)=ux+v$ and the ordering properties can be expressed in terms of addition and multiplication. Various examples and modifications of these laws are discussed.

Marshall Hall, Jr. (Columbus, Ohio)

4807:

Tallini, Giuseppe. Sui q-archi di un piano lineare finito di caratteristica $p=2$. Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. (8) 23 (1957), 242–245.

La nota presente porta un interessante contributo allo studio dei k-archi (cioè insiemi di k punti a tre a tre non allineati) di un piano lineare finito $S_{2,q}$ di caratteristica 2 ed ordine $q=2^h$. È ben noto che il massimo valore di k per cui esista un k-arco di $S_{2,q}$ è $q+2$ e che ogni $(q+1)$-arco di $S_{2,q}$ è incompleto, nel senso che adesso si può aggiungere un punto in guisa da ottenere un $(q+2)$-arco. L'autore dimostra che in $S_{2,q}$ anche un q-arco è contenuto in un $(q+2)$-arco, estendendo così a qualsiasi valore di q un risultato ottenuto recentemente da B. Segre nel caso semplice $q=8$.

D. Gallarati (Genova)

4808:

Segre, Beniamino. Sui k-archi nei piani finiti di caratteristica due. Rev. Math. Pures Appl. **2** (1957), 289–300.

In a finite projective plane of order n there exist at most $n+1$ points, no three on a line, if n is odd. Such a set is called an oval, and Segre has shown [Canad. J. Math. **7** (1955), 414–416; MR **17**, 72] that in a plane over a finite field GF(p^r), p odd, an oval is necessarily an irreducible conic. If n is even, an oval of $n+1$ points will have a unique tangent through each point of the oval and all the tangents will pass through a single point, which the author calls the nucleus. For a plane over a finite field $GF(2^r)$, the $n+1$ points $(n=2^r)$ of an irreducible conic and the nucleus will form an oval of $n+2$ points. In this paper it is shown that for such planes, when $n=2$, 4, or 8, an oval of $n+2$ points consists of an irreducible conic and its nucleus. But for $n=2^r$, $r=5$ or $r\geqq 7$, there is a curve $y=x^s$ whose n finite points, together with two points on the line at infinity, also form an oval with $n+2$ points. Here $s=2^g$, where for r odd we may take $g=r-2$, while for even $r=2^a b\geqq 8$, we may take $g=b+2$.

Marshall Hall, Jr. (Columbus, Ohio)

4809:

Kustaanheimo, Paul. On the relation of congruence in finite geometries. Math. Scand. **5** (1957), 197–201.

This paper gives a simplified proof of the results of Järnefelt and Kustaanheimo [11te Skand. Matematikerkongress, Trondheim, 1949, pp. 166–182, Tanums Forlag, Oslo, 1952; MR **14**, 1008]. The axioms on congruence in a plane over GF(p^r), $p\neq 2$, imply that the distance between (x_1, y_1) and (x_2, y_2) is determined by the value of $(x_1-x_2)^2-k(y_1-y_2)^2$, where k is some non-square. The simplification rests on Segre's proof [Canad. J. Math. **7** (1955), 414–416; MR **17**, 72] that in such a plane a complete oval is necessarily an ellipse.

Marshall Hall, Jr. (Columbus, Ohio)

4810:

Hofmann, Ludwig. Über ein bei den Clifford'schen Flächen bestehendes Analogon des Satzes von Dandelin. Monatsh. Math. 62 (1958), 1–15.

It is a widely known fact that if a sphere touches a cone of revolution in a circle and a plane ε in a point P, then P is the focus of the conic of intersection of the plane ε and the cone. Not so well known is the fact that the theorem is valid if the cone be replaced by any quadric of revolution. This follows readily in complex euclidean space, as the author here observes, from the simple fact that if two surfaces φ and ψ touch in a curve c then the curves of intersection of these two surfaces with a plane ε touch in the points of intersection of ε with the curve c.

These results are here extended to complex non-euclidean space as follows. Theorem. If a plane ε intersects a Clifford surface ψ in no four generators of ψ and is not tangent to either the surface ψ or the absolute φ of the underlying metric, then four of the six foci of the conic of intersection of ψ and ε which belong to those principal axes of the curve which cut only one of the two axes of the surface ψ are points of tangency of the plane ε with four spheres each of which touches the surface ψ in a circle. In interpreting the theorem the following definitions should be kept in mind. A Clifford surface is a surface of the second degree which has in common with the absolute a skew quadrilateral of generators. Suppose l a conic and k the absolute in this plane, l and k not tangent. The common polar triangle of l and k is called the principal polar triangle of l, its vertices are the centers and its sides the principal axes of l. The six intersections of the four common tangents of l and k are the foci of l. A quadric which touches the absolute in a conic is a sphere and a conic which has double contact with its absolute is a circle.

The paper contains several specific examples illustrating the theorems. *L. M. Kelly* (East Lansing, Mich.)

4811:

Mihăileanu, N. Courbes et surfaces de Tzitzeica dans la géométrie noneuclidienne. Gaz. Mat. Fiz. Ser. A (N.S.) 10(63) (1958), 468–472. (Romanian. French and Russian summaries)

In analogy to the curves of Tzitzeica (Ţiţeica) in 3-dimensional non-euclidean space with absolute $q^2x_0^2+x_1^2+x_2^2+x_3^2=0$, for which $\tau \sin^2 d/q$ is constant (τ torsion, d distance from the origin to the osculating plane), "surfaces of Tzitzeica" can be defined as surfaces for which $k \sin^2 d/q$ is constant. Here k^{-2} is the total curvature and d the distance from the origin to the tangent plane. Both relations are invariant under the projective group which leaves the origin in place. The asymptotic lines on surfaces of Tzitzeica are themselves curves of Tzitzeica. *D. J. Struik* (Cambridge, Mass.)

CONVEX SETS AND DISTANCE GEOMETRIES

See also 4761.

4812:

Holladay, John C. Matrix Nim. Amer. Math. Monthly 65 (1958), 107–109.

A generalization of Nim is described in which the piles of counters are arranged in a rectangular array. Guy and Smith [Proc. Cambridge Philos. Soc. 52 (1956), 514–522; MR 18, 546] and Holladay [Contributions to the theory of games, vol. 3, pp. 189–200, Princeton Univ. Press, 1957; MR 20 #2236] have shown that optimal strategies based on safe and unsafe positions, as in Nim, exist. A criterion to describe the safe and unsafe positions is given. *E. D. Nering* (Tucson, Ariz.)

4813:

Vorob'ev, N. N. Reduced strategies for games in the generalized form. Dokl. Akad. Nauk SSSR (N.S.) 115 (1957), 855–857. (Russian)

The author extends results of H. W. Kuhn [Proc. Nat. Acad. Sci. U.S.A. 36 (1950), 570–576; and "Contributions to the theory of games", vol. 2, pp. 193–216, Princeton Univ. Press, 1953; MR 12, 515; 14, 999] on replacing mixed strategies of players with perfect recall by pure strategies. *A. Dvoretzky* (Jerusalem)

4814:

Heppes, Aladár; and Révész, Pál. A splitting problem of Borsuk. Mat. Lapok 7 (1956), 108–111. (Hungarian. Russian and English summaries)

"H. G. Eggleston [J. London Math. Soc. 30 (1955), 11–24; MR 16, 734] has proved for $n=3$ Borsuk's conjecture according to which every point-set in the n-space with diameter 1 can be split into $(n+1)$ disjunct subsets with diameters <1. The present paper contains a simpler proof for the special case when the point-set contains only a finite number of points." *Author's summary*

GENERAL TOPOLOGY, POINT SET THEORY

See also 4613, 4685, 4773, 4774, 4814.

4815:

Hayashi, Yoshiaki. Extension of a topological mapping. Math. Japon. 4 (1957), 207–212.

The author claims to prove that a homeomorphism between two compact totally disconnected subsets of E^n can always be extended to a homeomorphism of E^n onto itself. However, this is contradicted for $n=3$ by Antoine's famous example [J. Math. Pures Appl. 4 (1921), 221–325] of a Cantor set in E^3 whose complement fails to be simply connected, and for arbitrary $n\geqq 3$ by Blankinship's examples [Ann. of Math. (2) 53 (1951), 276–297; MR 12, 730]. The best available positive result in this direction appears to be the easily proved extension theorem for countable compact sets. [See, for example, the reviewer's proof in Ann. of Math. (2) 66 (1957), 454–460; MR 20 #1200.] *V. L. Klee, Jr.* (Copenhagen)

4816:

Whyburn, G. T. Sense and orientation on the disk. Amer. Math. Monthly 64 (1957), no. 8, part II, 103–106.

A simple, elementary approach is provided to known results (fundamental to topological and mapping theorems) regarding orientation and sense agreement for plane closed curves (or curves on a disk). These results are usually treated only "after quite elaborate and sophisticated techniques have been developed," and may be used to establish such basic results as orientability of 2-manifolds. *R. L. Wilder* (Ann Arbor, Mich.)

4817:

Rudin, M. E. A connected subset of the plane. Fund. Math. 46 (1958), 15–24.

Assuming the continuum hypothesis, it is shown that

there exists a non-degenerate connected set in the plane which is disconnected (even dispersed) by every one of its uncountable subsets. This answers a question proposed by Erdös in 1944 and is effected by constructing a certain indecomposable continuum I and then showing that the desired type of set exists as a subset M of I which contains at most one point from each composant of I.

G. T. Whyburn (Charlottesville, Va.)

4818:

Ward, L. E., Jr. On dendritic sets. Duke Math. J. **25** (1958), 505–513.

Let X denote a connected locally connected Hausdorff space; X is called dendritic if every two distinct $x, y \in X$ are separated by a point $z \in X$. A characterization of dendritic spaces is given by means of the existence of a partial order satisfying certain conditions [for an analoguous characterization of trees, i.e. compact dendritic spaces, cf. L. E. Ward, Proc. Amer. Math. Soc. **5** (1954), 992–994; MR **17**, 180]. A well known result [cf. G. T. Whyburn, Analytic topology, Amer. Math. Soc. Colloq. Publ., New York, 1942; MR **4**, 86; p. 88] on dendrites is generalized as follows: a locally arcwise connected separable X is dendritic if and only if it contains no closed simple curve. It is shown that a locally compact X is dendritic if and only if every continuum $Y \subset X$ is a tree. Finally, it is proved that every convex (convexity being defined in terms of a partial order) dendritic space can be compactified as a tree; the convexity is essential, as shown by an example. *M. Katětov* (Prague)

4819:

Hartmanis, Juris. On the lattice of topologies. Canad. J. Math. **10** (1958), 547–553.

The lattice of all topologies on the set S of $n>2$ elements is a complete point lattice $LT(S)$, which is simple. The set $LT_1(S)$ of all T_1-topologies is a complete sublattice of $LT(S)$, but not a point lattice. If S is finite, then $LT(S)$ is complemented; if S is infinite, then all automorphisms of $LT(S)$ and $LT_1(S)$ are induced by permutations of S. Various other related results are also proved.

G. Birkhoff (Cambridge, Mass.)

4820:

Mrówka, S. On the convergence of nets of sets. Fund. Math. **45** (1958), 237–246.

The author generalizes to nets of sets some classical theorems on sequences of sets. Thus, if A_n, $n \in D$=directed set, is a net of subsets of X, lim inf A_n, lim sup A_n, and lim A_n (topological convergence) are defined. A topology on the space 2^X of closed subsets of X is said to induce topological convergence if convergence in the space of subsets is equivalent to topological convergence. If X is either compact metric or locally compact separable metric, it is known that 2^X can be metrized to induce topological convergence. If X is locally compact the author defines the *lbc*-topology on 2^X which has as a basis all sets of the form $[U_1, \cdots, U_n; V_1, \cdots, V_k]=\{A \in 2^X | A \cap U_i \neq \emptyset \text{ and } A \cap \bar{V}_j=\emptyset \text{ for } i=1, \cdots, n;\ j=1, \cdots, k\}$, where U_i, V_j are open sets with compact closures. Th.: The *lbc*-topology induces topological convergence. Th.: If X is compact the *lbc*-topology coincides with the Vietoris topology. If X is not compact the Vietoris topology does not induce topological convergence. If X is not even locally compact no topology can induce topological convergence. *M. E. Shanks* (Lafayette, Ind.)

4821:

Mrówka, S. A generalization of a theorem concerning the power of a perfect compact metric space. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **6** (1958), 89–93.

Let m be an infinite cardinal. (1) If X is a locally compact space and each point of X is of character $\geqq m$, then $\bar{\bar{X}} \geqq 2^m$. (2) If X is a compact m-almost-metrizable space [Mrówka, Bull. Acad. Polon. Sci. Cl. III **5** (1957), 123–127; MR **19**, 299] and each point of X is of character m, then $\bar{\bar{X}}=2^m$. (3) [Cf. #4606 above.] If G is a locally compact group and the unit element is of character $\geqq m$, then $\bar{\bar{G}} \geqq 2^m$. (4) If G is a compact group and the unit element is of character m, then $\bar{\bar{G}}=2^m$.

L. Gillman (Princeton, N.J.)

4822:

Whyburn, G. T. Mappings on inverse sets. Duke Math. J. 23 (1956), 237–240.

Let f be a continuous mapping from a topological T_1-space X onto a topological T_1-space Y. Then f is said to be quasi-compact if the image of every open [closed] inverse set in X is open [closed] in Y. The kernel E of f is the set of all points x in X for which $x=f^{-1}fx$. An example is given to show that f may be quasi-compact yet $f|E$ may not. It is proved that if f is quasi-compact and B is a subset of Y having the property that any set B_0 in B with a limit point b in B contains a subset B_1 whose closure contains b and is contained in B_0+b, then $f|f^{-1}B$ is quasi-compact. If X and Y are weakly separable and one is Hausdorff, and if f is quasi-compact and has kernel E, then it is shown that $f|E$ is a homeomorphism. Finally, if f is quasi-compact and X_0 is an inverse set, there is defined a factorization $f=\phi_2\phi_1$ such that ϕ_1 is one-one on $X-X_0$, ϕ_2 is one-one on $\phi_1 X_0$, and both ϕ_1 and ϕ_2 are quasi-compact. *P. V. Reichelderfer* (Columbus, Ohio)

4823:

Weier, Josef. Über die Ausnahmepunkte von Abbildungen zweidimensionaler Mannigfaltigkeiten in die Kugelfläche. Arch. Math. 7 (1956), 374–376.

Let A be a two-dimensional finite euclidean manifold, B a twodimensional simplicial sphere, c a point in B, and f a continuous mapping from A into B. It is shown that there exists a mapping f^1 of A into B which is homotopic to f and which assumes the value c for at most one point in A. *P. V. Reichelderfer* (Columbus, Ohio)

4824:

Weier, Joseph. On the topological degree. Compositio Math. 13 (1958), 119–127.

For a positive integer n exceeding one, let ϕ_1, ϕ_2 be continuous maps from the closure of an open set U in Euclidean $(n+1)$-space E_{n+1} into n-space E_n such that the set A of points in $\bar{U}$ where ϕ_1 and ϕ_2 have the same value is a simplicial l-sphere in U. Using the natural orientations of the E_i, the author defines an integer which he terms the degree of A under (ϕ_1, ϕ_2). He shows that this degree is topologically invariant, is deformation invariant, and admits of a certain decomposition. He terms A unessential if for every open set U_1 in E_{n+1} which contains A and is contained in U there exist continuous maps f_i from $\bar{U}$ into E_n which agree with ϕ_i outside U_1 and are such that f_1 and f_2 have different values at every point. He shows that if A is unessential then the degree of A under (ϕ_1, ϕ_2) is zero. If the degree of A under (ϕ_1, ϕ_2) is zero, he proves that there exists a point a in U and continuous maps f_i from $\bar{U}$ into E_n which

agree with ϕ_i on the boundary of U and are such that a is the only point of $\bar{U}$ where f_1 and f_2 have the same value. He gives an example to show that the degree of A under (ϕ_1, ϕ_2) may be zero and yet the set A may be essential. Similar definitions are made and theorems are established for the situation where U is an open set in an $(n+1)$-dimensional finite Euclidean manifold and the ϕ_i are continuous maps from $\bar{U}$ into an n-dimensional Euclidean manifold. *P. V. Reichelderfer* (Columbus, Ohio)

ALGEBRAIC TOPOLOGY

See also 4502, 4588, 4823, 4824.

4825a:

Wu, Wen-tsün. On the realization of complexes in euclidean spaces. III. Acta Math. Sinica **8** (1958), 79–94. (Chinese. English summary)

4825b:

Wu, Wen-tsün. On the realization of complexes in euclidean spaces. III. Sci. Sinica **8** (1959), 133-150.

[Chinese and English versions of the same paper. For parts I and II, see Acta Math. Sinica **5** (1955), 505–552; **7** (1957), 79–101; Sci. Sinica **7** (1958), 251–297; MR **17**, 883; **20**#3536, #5471.] Let K be a finite simplicial complex and R^m the euclidean m-space. A topological map T from K into R^m is said to be a linear imbedding of K in R^m if T is linear on every simplex of K; T is said to be a semi-linear imbedding of K in R^m if it is a linear imbedding of some simplicial subdivision of K in R^m. In the first paper of this series a cohomology class $\Phi^m(K) \in H^m(K^*, I_m)$ was defined. The main theorem in the present paper states that there exists a semi-linear imbedding of K in R^{2n}, where $n=\dim(K)$ if and only if $\Phi^{2n}(K)=0$. *Sze-tsen Hu* (Detroit, Mich.)

4826:

✱Massey, W. S. Some higher order cohomology operations. Symposium internacional de topología algebraica [International symposium on algebraic topology], pp. 145–154. Universidad Nacional Autónoma de México and UNESCO, Mexico City, 1958. xiv+334 pp.

The "triple product" cohomology operation [Uehara and Massey, Algebraic geometry and topology, a symposium in honor of S. Lefschetz, Princeton Univ. Press, 1957, pp. 361–377; MR **19**, 974] is generalized to n-tuple products in the following fashion. Let K be an abstract cell complex. An abstract sheaf $\mathfrak{G}$ on K is defined to be a function which assigns to each simplex $\sigma \in K$ a group $G(\sigma)$ together with incidence homomorphisms $I(\sigma, \tau): G(\tau) \to G(\sigma)$ satisfying appropriate identities. Given a ring R, the canonical sheaf $\mathfrak{G}(R)$ consists of the $(p+2)$-fold tensor product of R for each p-simplex of K, with the incidence homomorphism for a simplex σ and its ith face σ^i given by $I(\sigma^i, \sigma)(r_0 \otimes \cdots \otimes r_{p+1}) = r_0 \otimes \cdots \otimes r_i r_{i+1} \otimes \cdots \otimes r_{p+1}$. Let X be a topological space and Γ an associative graded ring of cochains for X; for example, Γ might be the singular cochains. The set of chains $C(K; \mathfrak{G}(\Gamma))$ inherits a boundary operator from K and a coboundary operator from Γ, making it into a bi-complex. This bi-complex gives rise to a spectral sequence the successive d's of which are the new cohomology operations. The triple product is shown to be a special case of these operations. For each of these new operations a topological space is constructed on which it is non-zero. *E. H. Brown* (Waltham, Mass.)

4827:

✱Steenrod, N. E. Cohomology operations. Symposium internacional de topología algebraica [International symposium on algebraic topology], pp. 165–185. Universidad Nacional Autónoma de México and UNESCO, Mexico City, 1958. xiv+334 pp.

The author gives an exposition of some of the basic properties of primary cohomology operations. It is based on the work of the author, E. Thomas, Eilenberg and MacLane, Serre, and Cartan. He also proves the following reduction theorem for cohomology operations of several variables. Let C denote the family of cyclic groups each of whose orders is either infinite or a power of some prime. Then every cohomology operation whose coefficient groups are finitely generated can be expressed as a composition of the following ones: addition, coefficient homomorphisms, cup products with coefficient groups in C, and cohomology operations of one variable with coefficient groups in C. Also there is a discussion of the problem of finding generators and relations for the cohomology operations of one variable with coefficient groups in C.

F. P. Peterson (Cambridge, Mass.)

4828:

✱Dedecker, Paul. On the exact cohomology sequence of a space with coefficients in a nonabelian sheaf. Symposium internacional de topología algebraica [International symposium on algebraic topology], pp. 309–322. Universidad Nacional Autónoma de México and UNESCO, Mexico City, 1958. xiv+334 pp.

In Dedecker, Bull. Soc. Math. Belg. **6** (1953), 97–125 [MR **17**, 80] it is shown that the equivalence classes of principal fibre bundles with base B and structural group G are in 1-1 correspondence with the elements of $H^1(B; \mathfrak{G})$, where $\mathfrak{G}$ is the sheaf of germs of maps of B into G. Using fibre bundle constructions the author defined [Colloque de topologie algébrique, Louvain, 1956, George Thone, Liège; Masson et Cie, Paris, 1957, pp. 135–149; MR **19**, 570] a groupoid $\mathfrak{H}^1(B; \mathfrak{G})$ which contains $H^1(B; \mathfrak{G})$. In the first part of the present paper $\mathfrak{H}^1(B; \mathfrak{G})$ is defined and some of its properties are given. In the second part, two-dimensional cohomology groupoids are defined and an exact sequence is developed.

Let G be a normal subgroup of H such that G has a local section in H and let $K=H/G$. There is an exact sequence Σ of sheaves: $e \to \mathfrak{G} \to \mathfrak{H} \to \mathfrak{K} \to e$. With respect to Σ, groupoids $\mathfrak{H}_\Sigma{}^q(B; \mathfrak{G})$, $q=0, 1, 2$ are defined and the following exact sequence is obtained: $e_1 \to \mathfrak{H}_\Sigma{}^0(B; \mathfrak{G}) \to \mathfrak{H}^0(B; \mathfrak{H}) \to \mathfrak{H}_\Sigma{}^0(B; \mathfrak{K}) \to \mathfrak{H}_\Sigma{}^1(B; \mathfrak{G}) \to \cdots \to \mathfrak{H}_\Sigma{}^2(B; \mathfrak{G}) \to e_2$ (e_1 and e_2 are the set of units of the appropriate groupoids, and exactness means: image=inverse image of the set of units). As the author observes, the definition of $\mathfrak{H}_\Sigma{}^2(B; \mathfrak{G})$ is not entirely satisfactory because it does not coincide with $H^2(B; \mathfrak{G})$ when $\mathfrak{G}$ is abelian.

E. H. Brown (Waltham, Mass.)

4829:

✱Fary, I. Spectral sequences of certain maps. Symposium internacional de topología algebraica [International symposium on algebraic topology], pp. 323–334. Universidad Nacional Autónoma de México and UNESCO, Mexico City, 1958. xiv+334 pp.

This paper summarizes two earlier papers of the author [Ann. of Math. (2) **63** (1956), 437–490; **65** (1957), 21–73; MR **17**, 1118; **18**, 822]. It contains definitions, the main results, and indications of some of the proofs.

E. H. Spanier (Princeton, N.J.)

4830:

O'Neill, Barrett. On the Leray isomorphism theorem. Proc. Amer. Math. Soc. 9 (1958), 460–462.

Dans la théorie des faisceaux sur un espace paracompact X, un théorème qui remonte essentiellement à Leray affirme que, sous certaines hypothèses, un homomorphisme $\varphi: A \to \bar{A}$ de faisceaux de groupes différentiels gradués définit un isomorphisme $H^n(\Gamma(A)) \to H^n(\Gamma(\bar{A}))$ des groupes d'homologie des groupes de sections $\Gamma(A)$ et $\Gamma(\bar{A})$. [Cf. H. Cartan, Séminaire de topologie algébrique de l'Ecole Normale Supérieure, 1950–51; MR 14, 670; Exposé 19, th. 4.] L'Auteur perfectionne ce résultat et démontre que sous les hypothèses: (1) $H^p(X, H^q(A)) \to H^p(X, H^q(\bar{A}))$ est surjectif pour $p+q=n$, et injectif pour $p+q=n+1$; (2) $H^q(H^p(X, A)) \to H^q(H^p(X, \bar{A}))$ est un isomorphisme pour $p \geqq 1$, q quelconque; (3) les degrés de A ou $\bar{A}$ sont bornés inférieurement, ou la dimension de X est finie; on peut conclure que $H^r(\Gamma(A)) \to H^r(\Gamma(\bar{A}))$ est surjectif pour $r=n$, et injectif pour $r=n+1$.

La démonstration utilise uniquement la technique des suites spectrales, et prouve en réalité un théorème valable chaque fois qu'on considère les deux suites spectrales déduites d'un complexe double. *H. Cartan* (Paris)

4831:

✶Kan, Daniel M. The Hurewicz theorem. Symposium internacional de topología algebraica [International symposium on algebraic topology], pp. 225–231. Universidad Nacional Autónoma de México and UNESCO, Mexico City, 1958. xiv+334 pp.

A proof is given, in a purely semisimplicial context, of the classical theorem of Hurewicz on the isomorphism of the first non-vanishing homology and homotopy groups.

Let G be a c.s.s. group (i.e., a complete semisimplicial complex in which the elements of each dimension are a group and the face and degeneracy operators homomorphisms). Define the subgroup $\bar{G}_n \subset G_n$ as the intersection of the kernels of all face operators other than the first; the first face operator induces a homomorphism $\bar{\partial}: \bar{G}_{n+1} \to \bar{G}_n$ such that $\bar{\partial}\bar{\partial}(\sigma)=1$, $\sigma \in \bar{G}$; hence $\bar{G}$ can be regarded as a complex. (This construction is due to J. C. Moore, who proceeds to define $\pi_n(G)=H_n(\bar{G})$.)

Let K be a c.s.s. complex with a single vertex. A c.s.s. group G is defined by letting G_n be the group generated by the elements of K_{n-1} with certain relations, and suitable face and degeneracy operators. (G corresponds to the loop-space of K in the geometric analogue.) The homotopy groups of K are now defined by

$$\pi_n(K)=H_{n-1}(\bar{G}).$$

After these preliminaries, the theorem of Hurewicz reduces to the following group-theoretic statement:

Let F be a free c.s.s. group, $B=F/[F, F]$ ($[F, F]$ is the commutator subgroup), $l: F \to B$ the natural map, inducing $\bar{l}: \bar{F} \to \bar{B}$. Then: (i) $\bar{l}_*: H_0(\bar{F}) \to H_0(\bar{B})$ is onto and has kernel $[H_0(\bar{F}), H_0(\bar{F})]$. (ii) Suppose $H_i(\bar{F})=0$ for $0 \leqq i < n$. Then $\bar{l}_*: H_n(\bar{F}) \to H_n(\bar{B})$ is an isomorphism and $\bar{l}_*: H_{n+1}(\bar{F}) \to H_{n+1}(\bar{B})$ is onto.

V. Gugenheim (Baltimore, Md.)

4832:

Postnikov, M. M. Cubical resolvents. Dokl. Akad. Nauk SSSR (N.S.) 118 (1958), 1085–1087. (Russian)

In this note the author transcribes the notion of a Postnikov system (=homotopy resolvent) into cubical homology. The author defines a cubical complex, following Kan. If G is a group, one may then define cochains of K with values of G and there is a well-defined notion of a 1-cocycle even if G is non-abelian. Moreover, if G is a G_1-module and $a^1 \in Z^1(K; G_1^*)$, one may define the nth cohomology group, $H^n_{a^1}(K; G)$ of K with values in G, with respect to the cocycle a^1. The cubical analogues of the Eilenberg-MacLane complexes $E(G, p)$, $K(G, p)$ are described and the author adapts to the cubical theory the notion of extending a complex K by means of a $(p+1)$-cocycle $k \in Z_{a^1}{}^{p+1}(K; G)$; this is just the fibre space over K with characteristic cocycle k. Such an extension is called a p-extension.

A sequence $K_1, K_2, \cdots$, of cubical complexes is called a resolvent sequence if $K_1=K(G_1, 1)$ for some group G_1 and if K, $i>1$, is an i-extension of K_{i-1} by a cocycle k_{i-1} of K_{i-1} with values in the G_1-module G_i, with respect to the fundamental 1-cocycle of $K(G_1, 1)$. Since $K_{i-1}{}^{i-1}=K_i{}^{i-1}$, $K_{i-1}{}^i \subset K_i{}^i$, we may define a cubical complex K by putting $K^i=K_i{}^i$: then K is called the limit complex of the sequence. The system $\{G_i, k_i\}$ of groups and factors is called the cubical resolvent of the complex K. The notions of homomorphism and isomorphism are defined for resolvents and the limit complexes of two resolvent sequences are isomorphic if and only if the corresponding resolvents are isomorphic. *P. J. Hilton* (Ithaca, N.Y.)

4833:

Postnikov, M. M. Limit complexes of cubic resolvents. Dokl. Akad. Nauk SSSR (N.S.) 119 (1958), 207–210. (Russian)

The main topic of this note is the characterization of those cubical complexes which appear as limit complexes of cubical resolvents (see preceding review). A cubical complex satisfying the Kan extension condition is called an E-complex. Then an E-complex is called an EN-complex if it has just one zero cube and if it satisfies the usual minimality condition that two distinct cubes, comparable in the usual sense, are non-homotopic; and a cubical complex K is isomorphic to the limit complex of a cubical resolvent if and only if it is an EN-complex. Since, in particular, a minimal subcomplex of a (connected) E-complex K is an EN-complex and any two are isomorphic, it follows that K determines a resolvent which completely specifies (in principle) all its homotopy properties. With any path-connected space X we may associate its cubical singular complex $Q(X)$ which is an E-complex and thus define the (singular) cubical resolvent of X; moreover, it is not difficult to show that every EN-complex is the minimal complex of some polyhedron X.

Finally the author takes up the algebraic analogue of the loop-space. He defines the notion of an Ω-pair of resolvents $\{G_i, k_i\}$, $\{H_i, l_i\}$ and, given a pair of resolvents, gives necessary and sufficient conditions on their limit complexes for them to be an Ω-pair. If $\{G_i, k_i\}$, $\{H_i, l_i\}$ is an Ω-pair and $\{H_i, l_i\}$ is the resolvent associated with some 1-connected space X, then $\{G_i, k_i\}$ is the resolvent associated with $\Omega(X)$.

{On p. 209, l. 37, ϕ_i should map G_i to H_{i+1}; on p. 210, l. 11, '$\sigma \varepsilon^j \omega$' should be '$\sigma \varepsilon^i \omega$'; on l. 15 '$\psi_1$' should be '$\psi_0$'.}

P. J. Hilton (Ithaca, N.Y.)

4834:

Chang, Su-cheng. On normal forms of homotopy type and homotopy groups of certain polyhedra. Acta Math. Sinica 8 (1958), 102–131. (Chinese. English summary)

The homotopy group $\pi_n(X)$ ($n \geqq 2$) of a given arcwise connected polyhedron X is an abelian group and hence may be considered as a direct sum of cyclic groups of prime power order. As a result, one may define a function

ϕ_n on powers of prime numbers such that $\phi_n(p^r)$ is the number of copies of the cyclic group of order p^r occurring in the group $\pi_n(X)$. Though $\phi_n(p^r)$ is theoretically determined if the space X is given, there is still no practicable method for computing such invariants. By means of the theory of proper isomorphisms, the notions of block invariants, relative block invariants, characteristic polynomials, characteristic coefficients, Φ_1-torsions and Φ_2-torsions are introduced in this paper to show that they constitute a complete and independent system of integer-valued homotopy invariants of simple A_n^3-polyhedra ($n>3$) besides Betti numbers and torsions. Here, an A_n^3-polyhedron is said to be simple if its cohomology groups H^{n+r} ($r=1, 2, 3$) may be written as a direct sum $H_1^{n+r}+H_2^{n+r}$, where H_1^{n+r} is the direct sum of $q_r(\geqq 0)$ cyclic groups of the same order, 2^{p_r}, and H_2^{n+r} is the direct sum of a free group (if any) and cyclic groups of odd prime power orders (if any).

Sze-tsen Hu (Detroit, Mich.)

4835:

James, I. M. Symmetric functions of several variables, whose range and domain is a sphere. Bol. Soc. Mat. Mexicana (2) **1** (1956), 85–88.

Let K be the cartesian product of m copies of S^n and let G be a transitive permutation group of degree m which operates on K by permuting the factors. If u projects K onto the orbit space of K under G, a map $f:K\to S^n$ is called G-invariant if it factors through u. The type of f is the degree of its restriction to S^n, embedded in K as the first factor. The author studies the problem of what types q occur for given n and G. Only the case of n odd is interesting, and then we may assume $m>1$ for non-triviality. We may also suppose that $n>1$ since every type occurs if $n=1$.

The author proves that the type q occurs if and only if q is a multiple of a certain positive integer k, none of whose prime factors is greater than m. As special — and important — cases the author considers: (i) m is prime and $G=Z_m$; (ii) $G=S_m$, the symmetric group. In case (i) k is a multiple of m; in case (ii) k is divisible by every prime which does not exceed m. *P. J. Hilton* (Ithaca, N.Y.)

4836:

Adem, José. On the cohomotopy exact couple. Bol. Soc. Mat. Mexicana (2) **1** (1956), 72–84. (Spanish)

The author gives a proof of an "unpublished result of J. Adem" [see, for example, Massey, Ann. of Math. (2) **57** (1953), 248–286, MR **14**, 1111]. Let λ: $\pi_n(S^q)\to\pi_{n+2}(S^{q+1})$ be defined by $\lambda(\alpha)=\eta\circ E(\alpha)$, where η is the non-zero element in $\pi_{n+2}(S^{q+1})$. Let $Sq^2:H^n(K;\pi_n(S^q))\to H^{n+2}(K;\pi_{n+2}(S^{q+1}))$ be the Steenrod square defined with respect to λ. Then the second differential in the cohomotopy exact couple [ibid.] can be identified with Sq^2. As an application, the author proves the following result. Let $\alpha\in\pi_m(S^p)$, $2\alpha=0$, $m<2p-3$, and let β be the non zero element in $\pi_{m+2}(S^m)$. Then $\alpha\circ\beta\equiv 0$ mod 2.

F. P. Peterson (Cambridge, Mass.)

4837:

✱Cartan, Henri; and Eilenberg, Samuel. Foundations of fibre bundles. Symposium internacional de toplogía algebraica [International symposium on algebraic topology], pp. 16–23. Universidad Nacional Autónoma de México and UNESCO, Mexico City, 1958. xiv+334 pp.

This paper is a clear, concise and self-contained exposition of local categories. A local category consists of a category $\mathscr{A}$ and a functor $L_{\mathscr{A}}:\mathscr{A}\to\mathscr{T}$, where $\mathscr{T}$ is a category of topological spaces. If A is an object of $\mathscr{A}$ and U is an open set in $L(A)$, then there is an object $A|U$ in $\mathscr{A}$, called the restriction of A to U. The main problem is to enlarge a local category $\mathscr{A}$ to a local category $\mathscr{B}$ so that the objects of $\mathscr{A}$ are restrictions of objects in $\mathscr{B}$. The authors give an explicit construction to solve this problem and prove various theorems about this construction. The results are applied to the category of formal bundles $(\mathscr{B},\mathscr{F})$, where $\mathscr{B}$ is a local category (the bases), $\mathscr{F}$ is a category (the fibres) and an object of $(\mathscr{B},\mathscr{F})$ is a pair (B,ϕ) with B in $\mathscr{B}$ and ϕ: $L_{\mathscr{B}}(B)\to\mathscr{F}$.

J. J. Kohn (Waltham, Mass.)

4838:

Curtis, M. L.; and Lashof, R. Homotopy equivalence of fiber bundles. Proc. Amer. Math. Soc. **9** (1958), 178–182.

A classification of fibre bundles by means of continuous homomorphisms of the space of loops of the base into the structure group has been given by R. Lashof [Ann. of Math. (2) **64** (1956), 436–446; MR **18**, 497]. In the present paper the authors give an analogous method of classifying fibre bundles according to fibre homotopy type [see A. Dold, Math. Z. **62** (1955), 111–136; MR **17**, 519]. In this classification theorem the function space of all homotopy self-equivalences of the fibre plays the role of the structure group. The proof of the theorem is based on a result of Dold [loc. cit.].

This classification theorem enables one to define characteristic classes of a fibre bundle which are invariants of the fibre homotopy type. These classes are in the cohomology ring of the loop space of the base instead of the base space itself. *W. S. Massey* (Providence, R.I.)

4839:

Weier, Joseph. Conséquences d'un résultat de M. J. P. Serre. C. R. Acad. Sci. Paris **247** (1958), 907–908.

Let $f:S^m\to S^n$ be a map. If to each $p\in S^m$ corresponds a vector $v(p)\neq 0$ at $f(p)$ tangent to S^n and if v is continuous, the author calls v a variation of f and asks for conditions under which a variation exists. He gives a geometric description of a homomorphism $\pi_m(S^n)\to\pi_m(S^{n-1})$ and asserts that a variation exists if and only if $[f]$ lies in the kernel of this homomorphism. {The reviewer remarks that the problem is simply that of lifting f into the Stiefel manifold $V_{n+1,2}$ and this is possible if and only if $\partial[f]=0$, where ∂ is the boundary homomorphism of the fibration $V_{n+1,2}/S^{n-1}=S^n$.}

The author discusses also the question of the existence of independent variations; this problem is, of course, also susceptible of a familiar interpretation.

P. J. Hilton (Ithaca, N.Y.)

4840:

Weier, Josef. Zur Topologie der Abbildungen dreidimensionaler Mannigfaltigkeiten. Monatsh. Math. **62** (1958), 163–172.

It is shown that if M and N are closed triangulated manifolds, $\dim M=\dim N+1>1$, $\varepsilon>0$, and $f, g:M\to N$, then g may be ε-deformed to g' so that $\{x\in M|f(x)=g'(x)\}$ is a finite polyhedron of dimension less than 2. Under the additional assumption that M and N are oriented, a 1-dimensional integral cycle on M associated with f and g is defined. No applications are made.

E. Dyer (Chicago, Ill.)

4841:

Weier, Joseph. Sulla sezione delle classi di trasformazioni continue. Rend. Mat. e Appl. (5) **17** (1958), 1–14.

Let P be an orientable $(n+1)$-dimensional manifold and Q an orientable n-dimensional manifold. If f, g are

maps from P to Q the set $A \subseteq P$ of coincidences of f and g is called by the author the geometric section of f and g. The author has shown [#4840 above] that if F, G are homotopy classes of maps $P \to Q$ then one may choose maps $f \in F$, $g \in G$ such that A is a polyhedron of dimension ≤ 1.

Let σ be an oriented 1-simplex of A, which we now suppose to be a 1-dimensional polyhedron, and let $p_0 \in \sigma$. One may find n-simplexes S and T of P and Q, respectively, such that $p_0 \in S$, $A \cap \bar{S} = p_0$ and $f(S) \cup g(S) \subset T$. We may define a map h: $\dot{S} \to \dot{T}$ by projecting $f(p)$ from $g(p)$ onto $\dot{T}$, $p \in \dot{S}$. By adopting appropriate orientation conventions we may thus assign to σ an integer coefficient γ, the degree of h. If σ_i runs through the 1-simplexes of A, the chain $\sum \gamma_i \sigma_i$ is called the algebraic section of f and g. It is a cycle and essentially independent of the triangulation of A.

The main result of this paper is that the homology class of the algebraic section depends only on F and G. If we drop the requirement of orientability on P and Q, we get the same result mod 2. *P. J. Hilton* (Ithaca, N.Y.)

4842:

Borel, Armand. The Poincaré duality in generalized manifolds. Michigan Math. J. **4** (1957), 227–239.

The author reproves, within the framework of sheaf theory, various results on generalised manifolds (n-gms) given in Wilder's "Topology of manifolds" [Amer. Math. Soc. Colloq. Publ., New York, 1949; MR **10**, 614]. Granted the apparatus of sheaves and of modern homology theory, not available at the writing of Wilder's book, the resulting proofs are more lucid, and sometimes more general, than their originals. The author works with a locally compact space X, using Alexander-Spanier cohomology groups $H^*(X, L)$, over a principal ideal ring L. Let $C_c{}^i(X, L)$ denote the Alexander-Spanier cochains with compact carriers, and form $C_*(X, L) = \sum_i \mathrm{Hom}(C_c{}^i(X, L), L)$. Let $H_i(C_*(X, L))$ denote the homology groups using the transpose of the coboundary operator in $C_*(X, L)$; and let F denote the sheaf associated with $C_*(X, L)$, whose stalks are F_x. Theorem: if X is locally compact, paracompact, and n-dimensional, and if $H_n(F_x) \approx L$, $H_i(F_x) = 0$ $(i \neq n)$ for all $x \in X$, then $H_{n-i}(C_*(X, L)) \approx H^i(X, H_n(F))$, $i = 0, \pm 1, \pm 2, \cdots$. If L is a field, the first group is $\mathrm{Hom}(H^{n-i}(X, H_n(F)))$, but otherwise is not known in general. {Compare this, however, with the reviewer's result that if X is locally compact and orientable and L is the integers, then $H_c{}^i(X, L)$ is the $(n-i)$th integral homology group with compact carriers [#4843 below; p. 369].} The author goes on to reprove duality theorems of Poincaré type for n-gms over a field, due to Wilder, in particular for Wilder's "infinite" groups $h(X)$. During the arguments, he obtains, for coefficients not necessarily a field, several results involving cohomological local connectivity ("clc"), of which the following, due to Wilder for fields, is a sample: If X is locally compact and finite dimensional, then X is clc in all dimensions, if and only if the local Betti numbers $p^r(x)$, $x \in X$, are everywhere not actually infinite. *H. B. Griffiths* (Bristol)

4843:

Griffiths, H. B. Locally trivial homology theories, and the Poincaré duality theorem. Bull. Amer. Math. Soc. **64** (1958), 367–370.

This is a research announcement preceding a forthcoming memoir. Two new results are stated with brief indication of proof. The first one is a Vietoris-type mapping theorem for singular homology. The second result is a Poincaré duality theorem for locally compact orientable generalized manifolds R. The theorem is stated for Čech theory and integer coefficients. The intervening homology group $H_{cq}(R)$ is defined as the direct limit of usual Čech homology groups $H_q(\mathrm{Cl}\, G)$, where G runs through the set of all open subsets of R, having a compact closure $\mathrm{Cl}\, G$. Earlier results obtained by similar methods were limited to coefficients in a field [see R. L. Wilder, Topology of manifolds, Amer. Math. Soc. Colloq. Publ., New York, 1949; MR **10**, 614; ch. VIII]. Another approach, by means of sheaf theory, had difficulties in identifying homology groups [see #4842 above].

Proofs of both results are based on the well-known method of full realizations of partial chain mappings [see, e.g., S. Lefschetz, Topics in topology, Princeton Univ. Press, 1942; MR **4**, 86; p. 81], which consists in extending chain mappings, by induction on dimension, using acyclicity-type assumptions. This method is fully formalized in the paper, thus resulting in a general theory; the results of above are then given as examples. The author also lists several other facts as further examples. These include in particular the de Rham theorem. *S. Mardešić* (Princeton, N.J.)

4844:

Plans, Antonio. Contribution to the homotopy of systems of knots. Rev. Mat. Hisp.-Amer. (4) **17** (1957), 224–237. (Spanish)

Milnor associated to any n-link $L = (l_1, \cdots, l_n)$ in euclidean space certain (non-independent) integers $\mu(i_1, \cdots, i_r)$, $1 \leq i_j \leq n$, $i_j \neq i_k$, $2 \leq r \leq n$, and showed that the residue classes $\mu(i_1, i_2, \cdots, i_r)$ mod Δ_r, where Δ_r denotes the greatest common divisor of the integers $\mu(j_1, \cdots, j_s)$, $s < r$, are homotopy invariants of L. [See Ann. of Math. (2) **59** (1954), 177–195; MR **17**, 70.] Such a link is called trivial if it is homotopic to $(l_1', \cdots, l_n')$, where $l_1'(C)$, $\cdots$, $l_n'(C)$ are points, and almost trivial if, for each $j = 1, \cdots, n$, the link $(l_1, \cdots, \hat{l}_j, \cdots, l_n)$ is trivial. The author defines L to be almost k trivial if, for each $1 \leq j_1 < j_2 < \cdots < j_k \leq n$, the link $(l_1, \cdots, \hat{l}_{j_1}, \cdots, \hat{l}_{j_2}, \cdots, \hat{l}_{j_k}, \cdots, l_n)$ is trivial. Milnor showed that if L is almost trivial (=almost[1] trivial) the integers $\mu(i_1, \cdots, i_r)$ are all zero for $r < n$ and that L is homotopically determined by the integers $\mu(i_1, \cdots, i_n)$. The author generalizes this to the following: If L is almost[2] trivial (so that $\mu(i_1, \cdots, i_r) = 0$ for $r < n-1$), then it is homotopically determined by the integers $\mu(i_1, \cdots, i_{n-1})$ and the residue classes $\mu(i_1, \cdots, i_n)$ mod Δ_n. The proof is an involved calculation and is marred by a number of typographical errors and unexplained notations. *R. H. Fox* (Princeton, N.J.)

4845:

Yajima, Takeshi; and Kinoshita, Shin'ichi. On the graphs of knots. Osaka Math. J. **9** (1957), 155–163.

Let π be a regular normed projection of a knot K on a 2-sphere S. The $n+2$ regions of $S - \pi$ are put into two classes $\mathfrak{A}$ (white) and $\mathfrak{B}$ (black) in the standard way, i.e., adjacent regions are colored oppositely. [Reidemeister, Knotentheorie, Springer, Berlin, 1932; Chelsea, New York, 1948.] The graph $g(\pi)$ of π was defined by Bankwitz [Math. Ann. **103** (1930), 145–161] and has been used by various authors. Its vertices correspond to the $\mathfrak{A}$-regions $A_1, \cdots, A_\alpha$ and its edges correspond to the double points $D_1, \cdots, D_n$ of π; the two $\mathfrak{A}$-regions that impinge on D_i correspond to the end-points of the edge that correspond to D_i; the ith edge is signed $+$ or $-$ according to the twist at D_i of the surface depicted by the $\mathfrak{A}$-regions. The

dual graph $g'(\pi)$ is defined the same way but using the $\mathfrak{B}$-regions $B_1, \cdots, B_\beta$ instead of the $\mathfrak{A}$-regions.

The authors translate the operations Ω_i [Reidemeister, loc. cit.] as well as some derived operations into operations O on $g(\pi)$ and O' on $g'(\pi)$, and they point out that $g(\pi)$ and $g'(\pi)$ are equivalent, under the operations O. They also indicate how to use the graphs $g(\pi)$ to construct "unions" [cf. review below] of knots and finally observe that if $g(\pi)$ and $g'(\pi)$ are isomorphic but with reversal of signs then K is amphicheiral (termed "amphibious" by the authors).

All of this applies, with minor modifications, to links of multiplicity greater than 1. *R. H. Fox* (Princeton, N.J.)

4846:

Kinoshita, Shin'ichi; and Terasaka, Hidetaka. On unions of knots. Osaka Math. J. **9** (1957), 131–153.

The unions and skew unions of two knots κ and κ' are generalizations by the authors of the well-known product $\kappa\cdot\kappa'$. They describe them in terms of the graphs of knots [cf. review above]: if (k) is a graph of κ and (k') is a graph of κ' and n is a non-negative integer, bring together to coincidence a vertex A of (k) and a vertex A' of (k') and connect another pair of vertices B of (k) and B' of (k') with n edges $e_1, \cdots, e_n$ attached with one and the same sign $\pm$. If n is even, the resulting graph is the graph of a knot κ'' called a union of κ and κ' with winding number $\pm n$; if n is odd, the resulting graph may be the graph of either a knot or a link of multiplicity 2 and the resulting knot or link κ'' is called a skew union of κ and κ' with winding number $\pm n$. It should be noted that the union or skew union is not uniquely defined by κ, κ' and n; it depends also on the choice of the vertices B and B' (which depend on the choice of projections of κ and κ').

A more intuitive description of κ'' is as follows: let $\kappa\cdot\kappa'$ be represented by a simple closed curve made up of arcs $\alpha = AA'$ and $\alpha' = A'A$; thus there is a 2-sphere Q that meets α in just the two points A and A', contains α in its interior and α' in its exterior, and if β is any arc joining A to A' on Q the simple closed curves $\beta+\alpha$ and $\beta+\alpha'$ represent κ and κ' respectively. Let Q^* be a 2-sphere and let γ and γ' be simple arcs spanning Q^* with end-points C, D and C', D', respectively, such that γ and γ' are the two braid strings of a braid in the interior of Q^* representing the element σ^n, where σ is a selected generator of the braid group $\mathfrak{Z}_2$ [cf. Artin, Abh. Math. Sem. Univ. Hamburg. **4** (1926), 47–72]. Let us assume that Q^* is so chosen that it meets $\alpha+\alpha'$ in just two arcs: $CD\subset\alpha$ and $C'D'\subset\alpha'$ if n is even, and $CD'\subset\alpha$ and $C'D\subset\alpha'$ if n is odd. Then κ'' is represented by $\alpha - CD + \gamma + (\alpha' - C'D' + \gamma')$ in the first case and by $(\alpha - CD') + (\alpha' - C'D) + \gamma + \gamma'$ in the second. In the second case κ'' is a link if, say, C and D' lie on α in the order $AD'CA'$ and C' and D lie on α' in the order $A'DC'A$.

For $n=0$ the union κ'' is uniquely determined and is just the product $\kappa\cdot\kappa'$. If an inversion of space with respect to Q transforms α into α' and α' into α (so that κ' is the mirror image of κ), then κ'' is called a symmetric union (or symmetric skew union) of κ. (A symmetric skew union is always a knot.)

Theorem 1: Every union of non-trivial knots is non-trivial. Theorem 2: The Alexander polynomial $\Delta''(x)$ of a symmetric union κ'' of κ is equal to the square $(\Delta(x))^2$ of the Alexander polynomial $\Delta(x)$ of κ. Theorem 3: The Alexander polynomial $\Delta''(x)$ of a symmetric skew union κ'' of κ does not depend on n but only on κ (and the choice of vertices B and B'; cf. the review below). Theorem 4: If κ'' is a symmetric skew union of κ, then $\Delta''(-1) = (\Delta(-1))^2$. Theorem 5: If κ'' is a union of κ and κ', and if the polynomials $\Delta(x)$ and $\Delta'(x)$ of κ and κ' are equal, then $\Delta(x)$ must be a factor of $\Delta''(x)$.

In § 4 the authors construct non-trivial symmetric unions of the trivial knot. One of these has only eleven crossings and is thus the simplest (in terms of number of crossings) of all known non-trivial knots with trivial Alexander polynomial. In terms of the structure of the group G of the knot, the triviality of the Alexander polynomial means that $[G, G]$ is perfect.

In an appendix the authors prove that if on a tame simple closed curve two points X and Y are chosen separating the curve into simple arcs α and β then there can always be found a simple arc γ spanning $\alpha+\beta$ with end points X and Y such that the simple closed curves $\alpha+\gamma$ and $\beta+\gamma$ are both of trivial type.

{The definition of (skew) union seems rather artificial. One cannot escape the feeling that the theorems about it can be generalized to some more natural class of operations.} *R. H. Fox* (Princeton, N.J.)

4847:

Hashizume, Yoko; and Hosokawa, Fujitsugu. On symmetric skew unions of knots. Proc. Japan Acad. **34** (1958), 87–91.

If κ' is a symmetric skew union of a knot κ, κ_1 a knot and κ_2 a link determined in a certain way by κ and the choice of vertices B and B' [cf. review above], then $\pm x^p\Delta_{\kappa_1}(x) = \phi(x)\cdot\phi(1/x)$, where $\phi(x) = \pm x^{p_1}\Delta_{\kappa_1}(x) \pm x^{p_2}(x-1)\Delta_{\kappa_2}(x, x)$. This supplements theorem 3 of the paper reviewed above.

As a corollary the authors prove that if κ is a link of multiplicity 2, then the polynomial $\Delta_\kappa(x, x)$ of κ has even degree. This is an immediate consequence of a result of Torres and Fox [Ann. of Math. (2) **59** (1954), 211–218; MR **15**, 979; Cor. 3], or of a remark of Alexander [Trans. Amer. Math. Soc. **30** (1928), 275–306; pp. 301–302].

R. H. Fox (Princeton, N.J.)

DIFFERENTIAL GEOMETRY, MANIFOLDS

See also 5100.

4848:

Vidal Abascal, Enrique. Present state, methods and problems of differential geometry. Rev. Mat. Hisp.-Amer. (4) **17** (1957), 161–170, 238–257, 299–312; **18** (1958), 28–70. (Spanish)

Continuation from same Rev. (4) **17** (1957), 38–58 [MR **19**, 58].

4849:

Popa, I. Une propriété caractéristique du cercle. Gaz. Mat. Fiz. Ser. A (N.S.) **10**(63) (1958), 491–494. (Romanian. French and Russian summaries)

"Si l'on cherche les courbes planes convexes pour lesquelles le rapport de la largeur par le rayon de courbure correspondant ait une valeur constante, on trouve que cette constante ne peut prendre que la valeur 2 et la courbe est un cercle." *Résumé de l'auteur*

4850:

Rachwał, T. L'ordre du contact d'une courbe régulière avec la sphère osculatrice. Ann. Polon. Math. **5** (1958), 33–43.

The author gives four sufficient conditions for the order

of contact of a regular curve C with the osculating sphere. It is assumed that the vector equation $r=r(s)$ of C, satisfies certain conditions (Z) and (Z*) in which $n\geqq 4$. Let $k_0^{(j)}$ and $t_0^{(j)}$ denote the derivatives of order j, with respect to the arc lengths, of the curvature k and torsion t of the curve C at a point $M_0:r_0=r(s_0)$. In this article, the second order determinants

$$W_{np}=k_0^{(n-2)}t_0^{(p-2)}-k_0^{(p-1)}t_0^{(n-3)},$$

for $3\leqq n-1\leqq p$, are important. Theorem 1 is that if the curve C satisfies the conditions (Z) and (Z*), then the order of contact q of C with its osculating sphere is not less than n. If $k_0^{(n-2)}\neq 0$, $t_0^{(n-3)}=0$, then $q=n$. If $t_0^{(n-3)}\neq 0$ and if $\beta\neq -k_0^{(n-2)}/k_0^2t_0^{(n-3)}$, where β is the second coordinate of the osculating sphere contained in the normal plane at M_0, then $q=n$. Theorem 2 states that if: (1) the assumptions (Z) and (Z)* are satisfied; (2) $t_0^{(n-3)}\neq 0$; (3) the center of the osculating sphere at M_0 relative to the normal plane at M_0 is $\alpha=1/k_0$, $\beta=-k_0^{(n-2)}/k_0^2t_0^{(n-3)}$; (4) $W_{np}=0$ for $p=n, \cdots, m$, where $n\leqq m\leqq 2n-5$; (5) $W_{n,m+1}\neq 0$; then $q=m+2$. Theorem 3 is that if the assumptions (1), (2), (3), of Theorem 2 are satisfied, and if (4) $W_{np}=0$, for $p=n, \cdots, 2n-4$, (5) $W_{n,2n-3}\neq 0$, then $q\geqq 2n-2$. Finally, Theorem 4 is that if the assumptions (1), (2), (3), of Theorem 2 are satisfied, and if $W_{np}=0$ for $p=n, \cdots, 2n-3$, then $q=2n-2$.

J. De Cicco (Chicago, Ill.)

4851:

Greenspan, Donald. Note on vertices in Euclidean 3-space. Amer. Math. Monthly **64** (1957), 731–733.

L'auteur se propose de démontrer qu'un point d'une courbe gauche où le rayon de la sphère osculatrice a une valeur extremum, n'est pas nécessairement celui où la sphère osculatrice a au moins cinq points en commun avec la courbe en question.

Remarquons à ce propos que la relation dont l'auteur fait usage à la page 733 de son article et qui se trouve dans "Lectures on classical differential geometry", par D. J. Struik [Addison-Wesley, Cambridge, Mass., 1950; MR 12, 127; p. 32, formule 8–10] ne caractérise pas les courbes sphériques, puisque les courbes à courbure constante (courbes de Monge) y satisfont également.

F. Şemin (Istanbul)

4852:

Wróbel, T. H. Geometrical interpretations of the curvatures of curves on a hypersurface in a Euclidean n-space. Biul. Wojskowej Akad. Tech. **18** (1955), 38–139. (Polish and Russian)

4853:

Vincensini, Paul. Sur deux problèmes relatifs à la déformation des surfaces. J. Math. Pures Appl. (9) **37** (1958), 329–342.

The two problems are the determination of pairs of applicable surfaces S, S': 1) such that the asymptotic net of S is transformed into a conjugate net of S' (this mapping is reciprocal); and 2) such that the normals at two corresponding points of S and S' form a constant angle. L. Bianchi has studied these problems with the aid of elliptic space [Rend. Accad. Lincei (3) **13** (1904), 6–17]; in the present paper the considerations are purely euclidean in character. Where Bianchi based his investigations of the first case on surfaces of constant curvature in elliptic space, and of the second case on the involutes of normal tangent congruences to a sphere, also in elliptic space, the present investigation leads to congruences of spheres of constant curvature of ordinary space in the first case, and to analogous involutes of a sphere in ordinary space in the second case.

D. J. Struik (Cambridge, Mass.)

4854:

Arghiriade, Emmanuel. Réseaux et congruences d'ordre supérieur. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **23** (1957), 246–250.

Com'è noto, un doppio sistema coniugato di specie p sopra una superficie F è definito da un'equazione della forma:

$$(1)\qquad \sum_{i=0}^{p}\sum_{j=0}^{1} a_{ij}x^{ij}=0 \quad \left(x^{ij}=\frac{\partial^{i+j}x}{\partial u^i\partial v^j}\right).$$

Se i punti $x(u, v)$ di F verificano la (1) le curve u, v formano su F una rete di Bompiani-Segre. Nella nota presente l'autore considera un "equazione generalizzata di Laplace d'ordine (p, q)":

$$L\equiv \sum_{i=0}^{p}\sum_{j=0}^{q} a_{ij}x^{ij}=0 \quad (a_{pq}\neq 0);$$

essa definisce una "rete generalizzata d'ordine (p, q)". L'autore comunica alcuni risultati che formeranno l'oggetto di un prossimo lavoro.

D. Gallarati (Genova)

4855:

Drăgilă, Pavel. Propriétés différentielles affines des réseaux. Proc. Amer. Math. Soc. **8** (1957), 1127–1133.

L'A. inizia il suo lavoro con alcune critiche a vecchi lavori di Petersen e Guichard, che si propone di giustificare completamente in un lavoro successivo [gli stessi Proc. **9** (1958), 189–200; MR **20** #2339]. Si passa poi allo studio affine delle reti coniugate (réseaux) su una superficie di S_3, e dà la seguente caratterizzazione per una rete coniugata: sia u, v una rete di curve su una superficie S descritta dal punto M — corrispondente al vettore [Ortsvektor] r; sia $M_1(r_1)$ il trasformato tangenziale di M secondo la curva u (v=cost.): $r_1=r+pr_u$; se e soltanto se il vettore tangente in M_1 alla curva v (della superficie trasformata) è parallelo al piano tangente alla superficie S in M, la rete u, v è coniugata. — Mediante considerazioni dello stesso tipo si caratterizzano le reti coniugate a invarianti uguali, o per le quali uno o entrambi gli invarianti sono nulli. — Si accenna poi ad alcuni problemi di natura proiettiva.

V. Dalla Volta (Roma)

4856:

Drăgilă, Pavel. Transport parallèle distancié. Bull. Sci. Math. (2) **82** (1958), 59–67.

A pair of surfaces S and $\bar{S}$, imbedded in a euclidean E_3, are each endowed with a (u, v)-coordinate net. By separately imposing distinct conditions involving the tangent planes of S and $\bar{S}$ at points with the same coordinates (u, v), and the tangent vectors of the coordinate curves at (u, v), $(u, v+dv)$ and $(u+du, v)$, the author obtains criteria according to which the coordinate net on S (or on $\bar{S}$) is conjugate, or asymptotic, or forms a Tchebychef net. Similarly, conditions which ensure that the curves u=const. are geodesics are established.

H. Rund (Durban)

4857:

Hämisch, Werner. Differentiale und Tensoren. Math. Ann. **135** (1958), 420–443.

Verf. stellt die Aufgabe, die grundlegenden Begriffe der Theorie der differenzierbaren Mannigfaltigkeiten in übersichtlicher und "natürlicher" Weise zu definieren, so daß sich die wichtigsten Eigenschaften ohne Umwege ergeben. — Eine Mannigfaltigkeit wird definiert als eine Menge P mit einem Atlas X, d.h. einer Menge von umkehrbaren Abbildungen ("Karten") x aus P in einen cartesischen

Raum R. Ist $\bar{x}\subseteq P$ der Argumentbereich von x, so sei $P=\bigcup_{x\in X}\bar{x}$. Als "cartesische" Räume werden beliebige endliche Produkte endlicher Potenzen des reellen Zahlkörpers P zugelassen.

1. Um Funktionen auf Mannigfaltigkeiten differenzieren zu können, muß man auf die Atlanten zurückgreifen. Daher behandelt Verf. zunächst die Differentiation für Abbildungen φ eines cartesischen Raumes R in einen cartesischen Raum S. Mit $\xi\in R$ und $\eta\in R$ wird

$$d_\eta{}^\xi\varphi=\lim_{\rho\to+0}\frac{\varphi(\xi+\rho\eta)-\varphi(\xi)}{\rho}\qquad(\rho\in \mathrm{P})$$

gesetzt. φ heißt differenzierbar in ξ, wenn der Limes existiert und linear in η ist. Durch $(d^\xi\varphi)\eta=d_\eta{}^\xi\varphi$ wird dann eine lineare Abbildung $d^\xi\varphi$ von R in S, das (gewöhnliche) Differential von φ in ξ definiert. Die Kettenregel lautet z.B. $d^\xi(\psi\varphi)=(d^{\varphi(\xi)}\psi)(d^\xi\varphi)$, wobei rechts die linearen Abbildungen nacheinander auszuführen sind. Mit Hilfe dieser eleganten Formulierung der Differentialrechnung "mehrerer Veränderlicher" werden dann die Mannigfaltigkeiten untersucht.

2. Eine Mannigfaltigkeit P mit einem Atlas X von Abbildungen x aus P in einen cartesischen Raum R, bei der die Kartentransformationen $x_1x_2{}^\vee$ (es sei stets $x^\vee$ die inverse Abbildung zu x) k-mal differenzierbar sind, heiße für $k\geqq 1$ differenzierbar. Die Topologie einer differenzierbaren Mannigfaltigkeit wird als die feinste Topologie eingeführt, bei der alle $x^\vee$ stetig sind. Um den "tangentialen Vektorraum" in $p\in P$ zu definieren, werden zunächst "Derivationen" in p eingeführt. Dazu sei f eine Abbildung aus P in einen cartesischen Raum S. Ist $fx^\vee$ für eine Karte x mit $p\in\bar{x}$ — und damit für jede solche Karte — differenzierbar in $x(p)$, so heißt f differenzierbar in p. Die linearen Abbildungen v^p des Moduls der in p differenzierbaren Abbildungen von P in S, die durch $v^p(f)=d_\eta{}^{x(p)}(fx^\vee)$ definiert werden, heißen Derivationen in p. Sie bilden einen zu R isomorphen Vektorraum, den "tangentialen" Vektorraum $\mathscr{V}^p$ der lokalen Vektoren in p. Vektoren auf P heißen die Abbildungen v von P, die jedes $p\in P$ auf einen lokalen Vektor v^p abbilden. Als Differential d^pf von f in p wird eine Abbildung des tangentialen Vektorraums $\mathscr{V}^p$ in S durch

$$(d^pf)v^p=v^p(f)$$

definiert. Bis auf die Isomorphie zwischen $\mathscr{V}^p$ und R fällt dieses Differential für den Fall $P=R$ mit dem gewöhnlichen Differential zusammen. Die üblichen Differentiationsregeln gelten auch hier noch.

3. Jetzt seien P bzw. Q Mannigfaltigkeiten mit den Atlanten X bzw. Y. f sei eine Abbildung von P in Q. $\mathscr{V}^p$ bzw. $\mathscr{W}^{f(p)}$ sei der tangentiale Vektorraum in $p\in P$ bzw. $f(p)\in Q$. f heißt differenzierbar, wenn für ein Kartenpaar x, y mit $p\in\bar{x}$ und $f(p)\in\bar{y}$ — und damit für jedes solche Paar — $yfx^\vee$ in $x(p)$ differenzierbar ist. Als "schwaches" Differential ∂^pf läßt sich eine Abbildung von $\mathscr{V}^p$ in $\mathscr{W}^{f(p)}$ definieren durch $(\partial^pf)v^p=w^{f(p)}$ mit $w^{f(p)}(g)=d_{v^p(yf)}{}^{yf(p)}(gy^\vee)$, denn $w^{f(p)}$ ist hierdurch — unabhängig von der Wahl eines y — eindeutig bestimmt. Für dieses schwache Differential gilt noch die Kettenregel. Da die Abbildungen in eine Mannigfaltigkeit i.a. aber keinen Modul bilden, werden die üblichen Differentiationsregeln sinnlos: das schwache Differential ist "gar kein wirkliches Differential".

4. Insbesondere für den Fall von Tensoren t wird man daher statt des schwachen Differentials ein in t lineares "tensorielles" Differential d^pt verlangen. Zur Definition der Tensoren schließt Verf. an seine frühere Arbeit [Math. Ann. **134** (1957), 101–113; MR **20** #1693] an. Man betrachte etwa den Modul F^p der reellwertigen in $p\in P$ differenzierbaren Funktionen und bilde endliche Produkte aus F^p und $\mathscr{V}^p$, z.B. $F^p\times\mathscr{V}^p\times F^p$. Ein lokaler Tensor t^p ist eine multilineare Abbildung eines solchen Produktes, die sich als Linearkombination von tensoriellen Produkten $v_1\odot v_2\odot v_3$ darstellen läßt mit $v_1\in\mathscr{V}^p$, $v_2{}^*\in\mathscr{V}^{p*}$ (dem Dual von $\mathscr{V}^p$, also dem Modul der linearen Abbildungen von $\mathscr{V}^p$ in P) und $v_3\in\mathscr{V}^p$. Diese tensoriellen Produkte sind als Abbildungen definiert durch

$$v_1\odot v_2{}^*\odot v_3(f_1, v_2, f_3)=v_1(f_1)\cdot v_2{}^*(v_2)\cdot v_3(f_3).$$

Die lokalen Tensoren bilden lokale Tensorräume, z.B. $T^p=\mathscr{V}^p\odot\mathscr{V}^{p*}\odot\mathscr{V}^p$. Die $v\in\mathscr{V}^p$ heißen auch lokale kovariante Vektoren, die $v^*\in\mathscr{V}^{p*}$ lokale kontravariante Vektoren. Die "Skalare" $f\in F^p$ heißen auch lokale Tensoren 0. Stufe. Neben dem tensoriellen Produkt $\odot$ von lokalen Tensoren werden "Kontraktionen" so eingeführt, daß z.B. für $t^p=v_1\odot v_2{}^*$ und $t'^p=v_3{}^*\odot v_4\odot v_5\odot v_6{}^*$ gilt

$$t^pt'^p=v_3{}^*(v_1)\cdot v_2{}^*(v_4)\cdot(v_5\odot v_6{}^*).$$

Den linearen Abbildungen l^p von lokalen Tensorräumen ineinander lassen sich eineindeutig lokale Tensoren t^p zuordnen, so daß $l^p(t'^p)=t^pt'^p$ gilt. Diese linearen Abbildungen lassen sich also eindeutig als lokale Tensoren darstellen. Tensoren sind Abbildungen, die jedem $p\in P$ einen lokalen Tensor t^p, etwa mit $t^p\in T^p$ zuordnen. Die Tensoren eines Typs sind also Abbildungen von P in $Q=\bigcup_p T^p$. Q läßt sich in naheliegender Weise zu einer differenzierbaren Mannigfaltigkeit machen. Statt des schwachen Differentials von Tensoren fordert Verf. von einem neu zu definierenden Differential $\mathfrak{d}^pt$, daß — wie für das (gewöhnliche) Differential — gilt: I. $\mathfrak{d}^pt$ ist linear in t; II. $\mathfrak{d}^pt$ ist eine lineare Abbildung von $\mathscr{V}^p$ in T^p. Jede solche lineare Abbildung ist eindeutig durch einen Tensor d^pt darstellbar, so daß $(d^pt)v^p=(\mathfrak{d}^pt)(\mathfrak{d}^p)$, wobei links die Tensoren zu kontrahieren sind. Durch $d_vpt=(d^pt)v$ entstehen "Derivationen", für die noch die Produktregel bzgl. $\odot$ und eine entsprechende Regel für die Kontraktion gefordert werden. Zu diesen Derivationen gehören die "kovarianten" Differentiale d^pt, die eindeutig durch n Tensoren aus $\mathscr{V}^p\odot\mathscr{V}^{p*}$ bestimmt sind. Das zu einer Metrik gehörige kovariante Differential ergibt sich als Spezialfall.

5. Verf. skizziert zum Schluß den Übergang von den Tensoren zu den antisymmetrischen Tensoren, für die sich eine alternierende Multiplikation definieren läßt und — vom tensoriellen Differential ausgehend — auch ein alternierendes Differential.

Das Ziel des Verf., daß "alle Begriffe aus einer möglichst simplen Formulierung des Mannigfaltigkeitsbegriffes mit einer inneren Notwendigkeit herausentwickelt werden ... und daher besonders reibungslose und leicht übersehbare Kalküle ermöglichen" wird so in der Tat weitgehend erreicht, nicht zuletzt aufgrund einer konsequent durchgeführten Notation, die allen Ansprüchen logischer Klarheit genügt. *P. Lorenzen* (Kiel)

4858:

Gołąb, S.; und Kucharzewski, M. Über den Begriff der Pseudogrössen. Tensor (N.S.) **8** (1958), 79–89.

The authors consider the transformation law of pseudo quantities. For example, a pseudo covariant vector transforms under coordinate transformation as follows: $v_\alpha=\tau A_\alpha{}^\lambda\bar{v}_\lambda$, where $A_\alpha{}^\lambda$ represent the first derivatives of the coordinate variables. The authors show that if τ de-

pends upon the coordinate transformation, then τ depends only on the signed powers of the determinant of the coordinate transformation. Thus the pseudo quantities are signed densities or capacities. The method of proof consists in showing: (1) If τ depends on the first and second derivatives of the coordinate transformation variables, then the second derivatives actually do not enter into the determination of τ; (2) by induction, it is shown that τ can not depend upon any derivatives but the first; (3) by use of a theorem of I. Schur [Über eine Klasse von Matrizen, die sich einer gegebenen Matrix zuordnen lassen, Dissertation, Berlin, 1901], it follows that τ depends only on the signed powers of the determinant of the coordinate transformation. *N. Coburn* (Ann Arbor, Mich.)

4859:

Şabac, Ion Gh. Quelques propriétés des champs biscalaires. Bul. Inst. Politehn Bucureşti **18** (1956), no. 3–4, 41–49. (Romanian. Russian and French summaries)

The author considers a vector field which is the gradient of a scalar or a scalar multiple of a gradient and hence normal to ∞^1 surfaces. A formula is obtained for the divergence of the curvature vector of the normal congruence in terms of the principal curvatures of the ∞^1 surfaces. Various relations involving the given vector field and the principal curvatures of the ∞^1 surfaces are determined. As the author notes, some of these relations are known. *N. Coburn* (Ann Arbor, Mich.)

4860:

Gutmann, M. Quelques formules stokiennes. Bul. Inst. Politehn. Bucureşti **18** (1956), no. 3–4, 119–128. (Romanian. Russian and French summaries)

By use of various vector identities, the general theorem of Stokes (relating line integrals to surface integrals and surface integrals to volume integrals) is expressed in different forms. Five formulas are derived for possible use in electromagnetic theory and fluid mechanics. Finally the variation of the circulation is considered.
N. Coburn (Ann Arbor, Mich.)

4861:

Kaul, S. K. A generalisation of the theorem of Green. Tensor (N.S.) **8** (1958), 8–13.

The author reviews the elements of extensor theory [see H. V. Craig, Vector and tensor analysis, McGraw-Hill Co., New York, 1943] and notes that if $T_j{}^i$ is a tensor then the extensor $T_{\alpha j}{}^{\alpha i}$ behaves as $T_j{}^i$ under coordinate transformation. By use of a theorem of Bochner [K. Yano and S. Bochner, Curvature and Betti numbers, Ann. of Math. Studies, no. 32, Princeton Univ. Press, 1953; MR **15**, 989; p. 31], a form of Green's theorem for extensors is obtained. *N. Coburn* (Ann Arbor, Mich.)

4862:

Abian, Smbat. On foundations of projective differential geometry. An. Şti. Univ. "Al. I. Cuza" Iaşi. Sect. I (N.S.) **3** (1957), 77–124. (Russian and Romanian summaries)

"The invariants of a system of homogeneous linear differential equations under the continuous group of projective transformations, characterize the projective properties of the integral manifold of that system. In this sense they constitute the foundation of the projective differential geometry of a class of projectively equivalent varieties. In this work, emphasis has been on (a) classification of invariants, (b) the number of functionally independent invariants, (c) special methods of obtaining explicit expressions of invariants, avoiding the tedious task of solving complete systems of partial differential equations." (Author's summary) *J. De Cicco* (Chicago, Ill.)

4863:

Picasso, Ettore. Invarianti proiettivo-differenziali di contatto di una superficie di S_4. Boll. Un. Mat. Ital. (3) **13** (1958), 160–172.

In questa nota l'autore considera una superficie σ dello spazio proiettivo S_4 e, in analogia a quanto fatto da E. Bompiani per le superficie dello spazio ordinario, definisce geometricamente mediante termini principali di birapporti infinitesimi le forme fondamentali ed alcuni elementi invarianti relativi a curve tracciate su σ. Per raggiungere lo scopo, associa a ciascun punto P di σ il sistema K delle quadriche che tagliano σ in linee aventi in P un punto quadruplo, ed in particolare due certe quadriche di carattere intrinseco, contenute in K, invarianti per collineazioni, le quali sostituiscono la quadrica di Lie utilizzata da E. Bompiani per le superficie di S_3.
D. Gallarati (Genova)

4864:

Speranza, Francesco. Sulle corrispondenze polari. Boll. Un. Mat. Ital. (3) **13** (1958), 157–159.

Sia $\mathscr{T}$ una trasformazione dualistica di S_r e cioè una corrispondenza tra punti ed iperpiani. In relazione all'intorno di primo ordine di una coppia generica A, a di elementi omologhi (per ipotesi non appartenentisi) si definisce una correlazione Γ nella stella di centro A nel modo seguente: se A' è un punto infinitamente vicino ad A ed a' l'iperpiano ad esso corrispondente in $\mathscr{T}$, sono omologhi in Γ la retta AA' e l'iperpiano che da A proietta l'$S_{r-2}\,aa'$. L'Autore comunica che se $\mathscr{T}$ è non singolare, Γ non può essere un sistema nullo (nella coppia generica di elementi corrispondenti); ne segue, per un risultato di E. Čech, che se Γ è involutoria $\mathscr{T}$ è necessariamente una corrispondenza polare. *D. Gallarati* (Genova)

4865:

Matsumoto, Makoto. A theorem for hypersurfaces of conformally flat space. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **29** (1955), 219–223.

The following theorem is proved: if p and only p $(0<p\leqq n)$ of the principal radii of a proper hypersurface V_n of a conformally flat space V_{n+1} are equal in a neighborhood U of V_n, then U contains ∞^{n-p} V_p such that V_p is umbilical in V_n and the conformal curvature tensor of V_p vanishes. This result is an immediate consequence of an earlier one by the reviewer [Ann. of Math. (2) **39** (1938), 762–786; in particular, Theorem 6.1 and Eq. (6.3)] upon noting that a conformally flat V_{n+1} is conformally equivalent to a space of constant curvature S_{n+1} and that the lines of curvature of a V_n in V_{n+1} are conformally invariant. *A. Fialkow* (Brooklyn, N.Y.)

4866:

Matsumoto, Makoto. Generalization of Lichnerowicz's theorem for a completely harmonic Riemannian space. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **29** (1955), 225–228.

A. Lichnerowicz [Bull. Soc. Math. France **72** (1944), 146–168; MR **7**, 80] proved that if a Riemann space H_n is completely harmonic and is imbedded isometrically as a hypersurface in a space S_{n+1} of constant curvature, then H_n is of constant curvature. Here H_n is restricted to have a positive definite first fundamental form. The author

proves this theorem under the weaker restriction on H_n that it be a proper hypersurface of S_{n+1}. His proof depends upon results by Walker [J. London Math. Soc. 24 (1949), 21–28; MR 10, 739] on the Ricci and Riemann curvature tensors of a completely harmonic space and a theorem of Fialkow [Ann. of Math. (2) 39 (1938), 762–785] on Einstein spaces which are hypersurfaces of an S_{n+1}. *A. Fialkow* (Brooklyn, N.Y.)

4867:

Sen, Hrishikes. On a certain conformal mapping of a non-simple K^*-space. Bull. Calcutta Math. Soc. 49 (1957), 193–197.

A non-flat Riemannian space $\bar{V}_n$ with metric tensor $\bar{g}_{ij}$ is called a K^*-space if its curvature tensor satisfies either $\bar{R}_{hijk,l}=\bar{R}_{hijk}\lambda_l$ or $\bar{R}_{hijk}\lambda_l+\bar{R}_{hikl}\lambda_j+\bar{R}_{hilj}\lambda_k=0$ and $\bar{R}_{hijk,l}=0$ for some non-zero vector λ_l. It is known that λ_l can be chosen to be a gradient, $\lambda_i=\partial\lambda/\partial x^i$. The author considers the conformal mapping V_n with metric tensor $g_{ij}=e^{-2\lambda}\bar{g}_{ij}$ on $\bar{V}_n$. Conditions are found under which V_n is also a K^*-space or under which the covariant derivative of its curvature tensor may be expressed simply in terms of $\bar{R}_{hijk}$ and λ_l. *A. Fialkow* (Brooklyn, N.Y.)

4868:

Sen, R. N. On a type of Riemannian space conformal to a flat space. J. Indian Math. Soc. (N.S.) 21 (1957), 105–114.

The author obtains the metric and curvature tensors of some of the conformally flat spaces which satisfy equations of the form

$$R_{hijk}=\phi[R_{hj}R_{ik}-R_{hk}R_{ij}]+\psi[g_{hj}g_{ik}-g_{hk}g_{ij}],$$

where R_{hijk}, R_{ij}, g_{ij} are the Riemann, Ricci and metric tensors, respectively. If these special spaces are Cartan symmetric, then either they are spaces of constant curvature or they are spaces considered in previous papers [same J. 17 (1953), 21–32; 19 (1955), 61–71; MR 14, 1123; 17, 781]. These latter spaces are Riemannian spaces which are compatible with pairs of teleparallelisms satisfying certain equations.

A. Fialkow (Brooklyn, N.Y.)

4869:

Sinyukov, N. S. Normal geodesic maps of Riemann spaces. Dokl. Akad. Nauk SSSR (N.S.) 111 (1956), 766–767. (Russian)

A geodesic mapping of a V_n on another $\bar{V}_n$, both Riemannian spaces, is normal if the V_n is laid out in two families of orthogonal surfaces S_m and S_{n-m} and these S_{n-m} are intersections of an m-orthogonal system of hypersurfaces, and, moreover, if this condition is preserved under the geodesic mapping, and this induces into the S_{n-m} a trivial, but into the S_m a non-trivial mapping. (If $\bar{g}_{ij,h}=2\varphi_h\bar{g}_{ij}+\varphi_i\bar{g}_{hj}+\varphi_j\bar{g}_{hi}$, then $\varphi_i=0$ is a trivial, $\varphi_i\neq 0$ a non-trivial mapping.) A theorem is derived concerning the existence of such normal geodesic mappings; forms of ds^2 and $d\bar{s}^2$ are given [cf. the author's paper, same Dokl. 98 (1954), 21–23; MR 16, 515].

D. J. Struik (Cambridge, Mass.)

4870a:

Ôtsuki, Tominosuke. Theory of affine connections of the space of tangent directions of a differentiable manifold. I, II, III. Math. J. Okayama Univ. 7 (1957), 1–74, 95–122.

4870b:

Ôtsuki, Tominosuke. Note on homotopies of some curves in tangent bundles. Math. J. Okayama Univ. 7 (1957), 191–194.

The author in these papers has as his goal the study of affine connections associated with a Finsler metric. Since these connections are in the tangent bundle over the base space, the author takes as his starting point a study of the affine connections in the tangent bundle. Now given an affine connection Γ it has associated with it two holonomy groups, the homogeneous holonomy group $\bar{H}$ and the general affine holonomy group $(A\bar{H})$. If instead of permitting all paths in the tangent bundle we restrict ourselves to paths that are the lifting of paths in the base space (point and tangent vector to the curve at the point) or lie in the fiber, we may define two new groups H and AH by parallel displacements along this new restricted class of paths. These are called restricted holonomy groups. The author then shows how to construct two new connections starting with Γ, call them Γ_1 and Γ_2, such that the homogeneous holonomy group of Γ_1 is isomorphic to H and the affine holonomy group of Γ_2 is isomorphic to AH. Part III of the paper #4870a applies these results to the study of affine connections associated with a Finsler metric.

In #4870b the author shows that one of the restrictions on homotopy of curves used in defining the restricted homogeneous holonomy group may be removed.

L. Auslander (Bloomington, Ind.)

4871:

Teodorescu, Ion D. Sur la classification des espaces $A_2(x, y)$ à connexion affine constante, localement euclidiens. An. Univ. "C.I. Parhon" Bucureşti. Ser. Şti. Nat. 7 (1958), no. 17, 23–27. (Romanian. Russian and French summaries)

Using the classification of G. Vranceanu-O. Borůvka [G. Vranceanu, Boll. Un. Mat. Ital. (3) 12 (1957), 145–153; MR 19, 764] the author proves that locally euclidean A_2 of constant $\Gamma_{jk}{}^i$ are of class three if and only if $\Gamma_{ik}{}^i=0$.

D. J. Struik (Cambridge, Mass.)

4872:

Vrănceanu, G. G. Sur les espaces à connexion affine localement euclidiens d'espèce trois. An. Univ. "C. I. Parhon" Bucureşti. Ser. Şti. Nat. 7 (1958), no. 17, 29–32. (Romanian. Russian and French summaries)

In the general case of point transformations of such A_2, which are of the third type, the problem reduces to the integration of a system of three partial differential equations in two variables. The integrability conditions lead to the result of O. Borůvka [Publ. Fac. Sci. Univ. Masaryk (1927), no. 85] that the connections of such A_2 depend, in general, on three arbitrary constants and four arbitrary functions of one variable.

D. J. Struik (Cambridge, Mass.)

4873:

Vrănceanu, G. G. Sur les transformations ponctuelles en deux variables, linéaires dans une des variables. An. Univ. "C. I. Parhon" Bucureşti. Ser. Şti. Nat. 7 (1958), no. 18, 19–22. (Romanian. Russian and French summaries)

The connection of A_2 under the condition that linear transformations of the form $u^1=\alpha x+\beta$, $u^2=ax+b$ exist, where α, β, a and b are functions of y, is studied. There exists one family of characteristic curves $\Gamma_{22}{}^1dy^3+(2\Gamma_{12}{}^1-\Gamma_{22}{}^2)dy^2dx-2\Gamma_{12}{}^2dydx^2=0$ parallel to the x-axis. If two such families exist of this form $dy=0$, then $\Gamma_{12}{}^2=0$ and the transformation can be written in the

form $u^1=\alpha x+\beta$, $u^2=b\neq 0$. Transformations of the third type (for which $2\Gamma_{12}{}^1=\Gamma_{22}{}^2$) can be written $u^1=b^{\frac12}x+\beta$, $u^2=b$. *D. J. Struik* (Cambridge, Mass.)

4874:

Lelong-Ferrand, Jacqueline. Sur les groupes à un paramètre de transformations des variétés différentiables. J. Math. Pures Appl. (9) 37 (1958), 269–278.

Let X be an infinitesimal transformation on a manifold V. Under what conditions does X generate a global group of transformations of V? The author gives a partial answer to this problem.

Let V be a manifold with a certain uniform structure such as a Riemannian metric. If the boundary of V (with respect to the given uniform structure) is sufficiently regular, the problem on V can be reduced, by the doubling process, to the one on the complete manifold $W=V\cup\partial V\cup V'$ where V' is a copy of V and ∂V stands for the common boundary of V and V'.

Assume now V to be complete. Let $r(x)$ be a non-negative function on V such that $\{x\in V; r(x)\leqq R\}$ is compact for every real number R (i.e., r is proper). Set $\theta_s(R=\sup_{r(x)=R}|Xr(x)|$. The first half of the main theorem states that if the integral $\int_R^\xi \theta_s(u)^{-1}du$ is divergent for all R, then X defines a global group of transformations of V. The second half states that, under some additional assumption, the converse is true.

As an application, the author proves that if X is an infinitesimal conformal transformation of a complete Riemannian manifold V and if the divergence of X is bounded then X generates a global group of conformal transformations of V. For the proof, let $r(x)$ be the distance between x and a fixed point x_0 of V and apply the main theorem. As a consequence, she obtains a new proof of the following theorem of the reviewer [Proc. Japan Acad. 30 (1954), 709–710; MR 16, 1053]: Every Killing vector field on a complete Riemannian manifold generates a global group of isometries. As another application of her main theorem, she gives a necessary and sufficient condition for an arbitrary X to generate a global group. The condition is not as practical as that of the main theorem.

Throughout the paper, the author assumes that the functions defining the change of local coordinates of V possess Lipschitz continuous derivatives of the first order and that tensor fields on V are Lipschitz continuous. Note that the function $r(x)$ on a Riemannian manifold (defined as the distance between x and x_0) is not differentiable but Lipschitz continuous.

S. Kobayashi (Cambridge, Mass.)

4875:

Hartman, Philip. On elliptic partial differential equations and uniqueness theorems for closed surfaces. J. Math. Mech. 7 (1958), 377–392.

The paper somewhat improves certain results of A. D. Alexandrov and H. Hopf. References to the extensive literature can be found in papers by A. D. Alexandrov [Vestnik Leningrad. Univ. 11 (1956), no. 19, 5–17; MR 19, 167] and H. Hopf [Convegno Internaz. Geom. Differenziale, Italia, 1953, pp. 45–54, Ed. Cremonese, Roma, 1954; MR 16, 167].

Let $F(H, K, u_1, u_2, u_3)$ be defined for $H>0$, $H^2\geqq K>0$, $|u|=[\sum u_i^2]^{\frac12}=1$, and satisfy (*) $(\partial F/\partial K_1)(\partial F/\partial K_2)>0$, where $2H=K_1+K_2$, $K=K_1K_2$. There is at most one closed surface S of class C^2 in E^3 with F of class C^1 and H, K as mean and Gauss curvature and $F=0$ at the point with inward normal n. (Under weak hypotheses and in E^n this result is found in Alexandrov, loc. cit., which came to the author's attention after submission of the paper.) If S is a local surface of class C^2 satisfying $F(H, K)=0$, but (*) only when $H^2=K$ (F of class C^1), then either all points of S are umbilics, or the umbilics are isolated and have negative index.

If F is of class C^1 for $H^2\geqq K$ and satisfies (*) and S is a closed surface without self intersections with $F=0$, then S is a sphere. The proofs rest on theorems on elliptic partial differential equations which are too involved to be stated here. *H. Busemann* (Cambridge, Mass.)

4876:

Kuiper, N. H. On some algebraic isometric imbeddings. Simon Stevin 32 (1958), 23–28.

It is observed that the mapping $u\to u\otimes\cdots\otimes u$ of a linear space into its k-fold Kronecker product maps the unit sphere into the unit sphere, and multiplies intrinsic distances by k. Consequently, this map yields isometric mappings of spheres and related manifolds into corresponding manifolds of higher dimensions. In this way, various isometric mappings defined by Blanusa and by Calabi are rederived and generalized, and a number of curious imbedded curves and surfaces exhibited.

J. T. Schwartz (New York, N.Y.)

4877:

✱Whitney, Hassler. Singularities of mappings of Euclidean spaces. Symposium internacional de topología algebraica [International symposium on algebraic topology], pp. 285–301. Universidad Nacional Autónoma de México and UNESCO, Mexico City, 1958. xiv+334 pp.

Soit f une application r fois différentiable de l'espace euclidien R^n dans R^m, $f^{(r)}$ la dérivée d'ordre r de f, application de R^n dans l'espace $\mathscr{L}^r$ des "jets" d'ordre r; on définit d'abord l'ensemble S_k des points de R^n, où le "corang" (deficiency) de f est strictement égal à k, dont on fait ensuite une étude locale comme ensemble algébrique réel dans $\mathscr{L}^1$. Les singularités d'ordre p sont liées à certains ensembles algébriques de l'espace $\mathscr{L}^p$, qu'on peut définir par induction sur p de la manière suivante: si Y est une sous-variété sans singularités partie d'un tel ensemble de $\mathscr{L}^p$, on ne considérera que les applications f telles que $f^{(p)}$ soit transversale sur Y; le plan tangent à l'image réciproque $(f^{(p)})^{-1}(Y)$ est caractérisé par la donnée en ce point de $f^{(p+1)}$; en écrivant que ce plan tangent occupe vis-à-vis du noyau N de $f^{(1)}$ et des cônes tangents aux ensembles critiques d'ordre $\leqq p$ une position déterminée, on définit une subdivision en ensembles algébriques (manifold collection) de l'ensemble image réciproque de Y dans $\mathscr{L}^{p+1}$. Pour qu'une application soit "localement générique", il faut et il suffit que $f^{(p)}(x)$ soit transversale aux ensembles critiques qu'elle rencontre pour tout p, et tout x d'un voisinage $U\subset R^n$. L'article contient une description explicite des singularités génériques pour $m, n\leqq 5$. Il se termine par l'énoncé du problème central de la théorie: une application localement générique est-elle stable au sens topologique?: existe-t-il des homéomorphismes φ de R^m, ψ de R^n telle que toute application g assez voisine de f (pour la C^r topologie) s'écrive: $g=\varphi f\psi$?

R. Thom (Strasbourg)

4878:

✱Eells, James, Jr. On the geometry of function spaces. Symposium internacional de topología algebraica [International symposium on algebraic topology], pp. 303–308. Universidad Nacional Autónoma de México and UNESCO, Mexico City, 1958. xiv+334 pp.

Un espace topologique M est une "variété généralisée

de modèle E'', où E est un espace vectoriel localement convexe complet, si tout point x de M possède un voisinage U et un homéomorphisme θ de U dans E, avec la condition usuelle de recollement différentiable des cartes sur l'intersection de deux ouverts U. Ceci permet de définir le fibré des vecteurs tangents, les 1-formes, et, par produit tensoriel, les p-formes sur M. Par exemple, si M est une variété riemannienne au sens usuel, l'espace vectoriel $L(K, M)$ des applications d'un compact K dans M peut être muni d'une structure de variété généralisée localement de Banach, avec norme de Finsler, à l'aide des segments géodésiques sur M. Le théorème de de Rham pourra s'établir sur ces variétés généralisées, dès qu'on a sur elles des partitions différentiables de l'unité, condition apparemment fort restrictive, que l'auteur pense éviter en faisant appel à la théorie des formes lipschitziennes dûe à Whitney. Exemple: L'espace des lacets de classe C^1 d'une variété riemannienne M, à dérivées de carré sommable, peut être muni d'une structure de variété localement hilbertienne. *R. Thom* (Strasbourg)

4879:

Nasu, Yasuo. On asymptotes on a 2-dimensional metric space. Tensor (N.S.) **7** (1957), 173–184.

With the notation of the reviewer's "The geometry of geodesics" [Academic Press, New York, 1955; MR **17**, 779], consider a G-space R. Let σ be a co-ray to the ray ρ. The totality of all co-rays to ρ is an asymptote to ρ. It is either a straight line or a ray, whose origin is called asymptotic conjugate to ρ. In a previous paper [Tôhoku Math. J. (2) **7** (1955), 157–165; MR **17**, 1124] the author began a systematic investigation of the set $K(\rho)$ of all asymptotic conjugate points to ρ. It was continued in other papers; the present one concentrates on the case where R is two-dimensional and hence a manifold. If $K(\rho)$ is closed then it is either empty or one point in an at most countable union of curves. From any point which is not an endpoint of these curves there are at least two asymptotes to ρ. If every asymptote to ρ is a ray, then $K(\rho)$ is connected. If R has non-positive curvature (in the reviewer's sense) then a neighborhood of an interior point of an asymptote to ρ is simply covered by asymptotes to ρ, on which the asymptote relation is symmetric and transitive.

If $K(\rho)$ is closed and $a \in K(\rho)$ then (for non-positive curvature) a neighborhood of a on $K(\rho)$ consists of a finite number of arcs which are disjoint except for a, and each of the angular domains determined (locally) by $K(\rho)$ contains part of exactly one asymptote to ρ with origin in a. *H. Busemann* (Cambridge, Mass.)

PROBABILITY

See also 4511, 4624, 4761, 4775, 4941, 4996.

4880:

Rubinštein, G. Š.; and Urbanik, K. Solution of an extremal problem. Teor. Veroyatnost. i Primenen. **2** (1957), 375–377. (Russian. English summary)

Let N be a set of pairs of integers (i, j) $(i, j=1, 2, \cdots, n)$, M a non-empty subset of N, and $\mathscr{P}_M$ the class of all systems $P=(p_{ij})$ $((i, j) \in N)$ where $p_{ij}\geqq 0$ for $(i, j) \in N$, $p_{ij}=0$ for $(i, j) \in N-M$ and $\sum_{(i,j)\in M} p_{ij}=1$. Define

$$\Phi(P)=\sum_{(i,j)\in N} p_{ij} \log\Big[p_{ij}\Big(\sum_{k=1}^n p_{ik} \sum_{l=1}^n p_{lj}\Big)^{-1}\Big].$$

The authors solve the problem (posed by A. N. Kolmogorov) of finding $\max_{P\in\mathscr{P}_M} \Phi(P)$. The latter turns out to be equal to $\log r(M)$, where $r(M)$ is the greatest number of pairs $(i_1, j_1), \cdots, (i_s, j_s)$ belonging to M and such that $i_k\neq i_l$, $j_k\neq j_l$ for $k\neq l$ $(k, l=1, \cdots, s)$.

G. Kallianpur (East Lansing, Mich.)

4881:

Weiss, Irving. Limiting distributions in some occupancy problems. Ann. Math. Statist. **29** (1958), 878–884.

The occupancy problem regards the random distribution of a number r of objects in a number N of cells. More than one object may be placed in one cell. It is shown that, when $\alpha=r/N$ $(\alpha>0)$, the asymptotic distribution of the number of unoccupied cells is normal as N, r approach infinity. *P. Johansen* (Copenhagen)

4882:

Sazonov, V. On characteristic functionals. Teor. Veroyatnost. i Primenen. **3** (1958), 201–205. (Russian. English summary)

Let $e_1, e_2, \cdots$ be an orthormal base on a separable Hilbert space H. Let A be S-operators, that is, a linear symmetric, non-negative and completely continuous operator with finite trace. Let J be the topology such that the class of finite intersections of sets $\{f \in H:(Af, f)^{\frac{1}{2}}<\lambda\}$, $\lambda>0$, is a base of neighborhoods of the origin θ. The author proves the following two theorems. I. A functional $\chi(t)$ on H is a characteristic functional of a probability distribution in H if and only if: (i) it is positive-definite and $\chi(\theta)=1$; (ii) it is continuous in norm; (iii) $\sum a_{nn}<\infty$, where $a_{nn}=\int x^2dF_n(x)$ and $F_n(x)$ is a distribution function in R' with characteristic function $\varphi_n(t)=\chi(te_n)$. II. A functional $\chi(t)$ on H is a characteristic functional of a probability distribution if and only if: (i) it is positive-definite and $\chi(\theta)=1$; (ii) $\chi(f)$ is continuous at θ in the J-topology. *M. Loève* (Berkeley, Calif.)

4883:

Linnik, Yu. V. General theorems on the factorization of infinitely divisible laws. Teor. Veroyatnost. i Primenen. **3** (1958), 3–40. (Russian. English summary)

The logarithm of the characteristic function of an infinitely divisible (i.d.) law admits, according to P. Lévy, a unique representation

$$\beta it-\gamma t^2+\int_{-\infty}^0\Big(e^{itu}-1-\frac{itu}{1+u^2}\Big)\,dG_-(u) + \int_0^\infty\Big(e^{itu}-1-\frac{itu}{1+u^2}\Big)\,dG_+(u)$$

with β real, $\gamma\geqq 0$ (the Gaussian part), and $G_-(u)$ and $G_+(u)$ (the negative and positive parts of the Poissonian spectrum (P.s.)) are non decreasing functions satisfying $G_-(-\infty)G_+(\infty)=0$, and $\int_{-\infty}^0 u^2dG_-(u)+\int_0^\infty u^2dG_+(u)<\infty$ for every $a>0$ and tends to zero as $a\to 0$. The P.s. is countable (finite or infinite) if there exist sequences of positive numbers μ_n, ν_n (called Poissonian frequencies) and non-negative numbers λ_n, λ_{-n} (called energy parameters) such that $G_-(u)=\sum_{-\nu_n\leq u} \lambda_{-n}$ and $G_+(u)=-\sum_{\mu_n>u} \lambda_n$. The present paper is devoted to the proof of the following very comprehensive theorem. A necessary condition that an i.d. law with $\gamma>0$ be decomposable only into i.d. components is that its P.s. be countable and that, moreover, there exist μ, ν and sequences k_n, k_{-n}, l_n, l_{-n} $(n=1, 2, \cdots)$ of natural numbers $\geqq 2$ such that $\{\mu_n\}\subset\{\cdots, k_{-1}k_{-2}\mu, k_{-1}\mu, \mu, \mu/k_1, \mu/k_1k_2, \cdots\}$ and $\{\nu_n\}\subset\{\cdots, l_{-1}l_{-2}\nu, l_{-1}\nu, \nu, \nu/l_1, \nu/l_1l_2, \cdots\}$. Furthermore, if the

P.s. is bounded, this necessary condition is also sufficient. The condition $\gamma>0$ is essential. Though the assumption, in the last part of the theorem, that the P.s. is bounded may be relaxed it is not known whether it can be dropped altogether. The intricate proof is a development of the author's method in previous papers. Further theorems are stated, but they will be dealt with in future installments of this fundamental study.

A. Dvoretzky (Jerusalem)

4884:

Mauldon, J. G. Characterizing properties of statistical distributions. Quart. J. Math. Oxford Ser. (2) **7** (1956), 155–160.

Let x_i $(i=1, 2)$ be independent identically distributed random variables with probability density $f(x)$. The author studies those $f(x)$ for which $(x_1+x_2)/|x_1-x_2|$ has the t distribution. From the general solution it follows, in particular, that x_i need not be normally distributed. Similarly, if x_1/x_2 has an F distribution, it does not follow that the x_i have a χ^2 distribution. The author relies heavily on results of F. M. Goodspeed [Canad. J. Math. **2** (1950), 223–237; MR **11**, 661]. [See also #4901 below.]

A. Dvoretzky (Jerusalem)

4885:

Blum, J. R. A note on stochastic approximation. Proc. Amer. Math. Soc. **9** (1958), 404–407.

Let (X_n) and (Y_n) be infinite sequences of random variables. Denote by $I\{\cdots\}$ the indicator of the set described and by $U^+[U^-]$ the positive [negative] part of U. The author proves that if (1) $X_n-\sum_{j=1}^n Y_j$ converges with probability one (w.p. 1), (2) for some integer k and every $\varepsilon>0$, $\sum_{n=k}^\infty I\{X_n\geq\varepsilon, \cdots, X_{n-k+1}\geq\varepsilon\}Y_n{}^+<\infty$ and $\sum_{n=k}^\infty I\{X_n\leq-\varepsilon, \cdots, X_{n-k+1}\leq-\varepsilon\}Y_n{}^->-\infty$ w.p. 1; then X_n converges w.p. 1 if and only if $Y_n\to0$ w.p. 1. He then applies this result to get convergence w.p. 1 in a generalized version of the Robbins-Munro stochastic approximation scheme for locating the root of a regression equation.

A. Dvoretzky (Jerusalem)

4886:

Sacks, Jerome. Asymptotic distribution of stochastic approximation procedures. Ann. Math. Statist. **29** (1958), 373–405.

The paper studies the asymptotic behavior of the Robbins-Munro and Kiefer-Wolfowitz procedures. We reproduce only the result concerning the first. Let: (1) $M(x)$ be Borel-measurable, $(x-\theta)(M(x)-\alpha)\geq0$ and $|M(x)-\alpha|<K|x-\theta|$ for all x, $M(x)=\alpha+\alpha_1(x-\theta)+o(|x-\theta|)$ as $x\to\theta$, with $\alpha_1>0$ and $\inf|M(x)-\alpha|>0$ over every compact interval not containing θ; (2) $Z(x)$ be, for every x, a random variable, $\sup_x Z(x)<\infty$, $EZ^2(x)\to\sigma^2$ as $x\to\theta$ and $\lim_{R\to\infty}\lim_{\varepsilon\to0}\sup_{|x-\theta|<\varepsilon}\int_{\{|Z(x)|>R\}} Z^2(x)dP=0$; (3) $a_n=A/n$ with $A>1/(2\alpha_1)$. Then if $EX_1{}^2<\infty$ and $X_{n+1}=X_n-a_n[M(x_n)-\alpha+Z(x_n)]$, it follows that $n^{\frac{1}{2}}(X_n-\theta)$ is asymptotically normally distributed with mean 0 and variance $A^2\sigma^2/(2A\alpha_1-1)$. The first results of this nature were proved by K. L. Chung [same Ann. **25** (1954), 463–483; MR **16**, 272]. Chung and his followers used the method of moments; the author utilizes instead a central limit theorem for dependent random variables and this enables him to obtain asymptotic normality under more general conditions than hitherto. Some remarks on "best possible" aspects of the results are made and the multi-dimensional case is also treated.

A. Dvoretzky (Jerusalem)

4887:

Prohorov, Yu. V. Strong stability of sums and infinitely divisible laws. Teor. Veroyatnost. i Primenen. **3** (1958), 153–165. (Russian. English summary)

In 1950 the author published the first necessary and sufficient conditions for the strong law of large numbers (SLLN) and deduced sufficient conditions, known and new [Izv. Akad. Nauk SSSR. Ser. Mat. **14** (1950), 523–536; MR **12**, 425]. In the present paper, he gives new sufficient conditions which contain all the preceding ones and which, in a sense, are also necessary: Without loss of generality one may assume that the independent summands $\xi_1, \xi_2, \cdots$ with d.f.'s $F_1, F_2, \cdots$ are symmetrically distributed and that $|\xi_n|<n$. Then the SLLN holds if $\sum_{r=0}^\infty P[\tau_r\geq\varepsilon]<\infty$ for every $\varepsilon>0$, where the τ_r are infinitely decomposable random variables with characteristic functions whose logs are $\psi(t)=2^r\int(e^{itx/2^r}-1)d\bar{F}_r(x)$, $\bar{F}_r=\sum_{\kappa=2^r+1}^{2^{r+1}} F_\kappa$. This condition is also necessary when the ξ are identically distributed and is also necessary in order that all sequences with given $F_1, F_2, \cdots$ obey the SLLN. Under some regularity assumptions, necessary and sufficient conditions are obtained for random variables ξ_n with $\mathrm{P}(\xi_n=\pm a_n)=p_n/2$, $\mathrm{P}(\xi_n=0)=1-p_n$. Various new sufficient conditions are given.

M. Loève (Berkeley, Calif.)

4888:

Lamperti, John. Some limit theorems for stochastic processes. J. Math. Mech. **7** (1958), 433–448.

Let X_n, $n=0, 1, \cdots$ be a Markov chain with stationary transition probabilities. Let Y_n be the time since the last visit to a measurable set A of states visited infinitely often with probability one if $X_0=x_0\in A$. It is shown that under various regularity conditions $\lim_{n\to\infty}\Pr(Y_n/n\leq t)=F_\alpha(t)$, where F_α is a distribution function on $[0, t]$, if and only if $\lim_{n\to\infty}E(Y_n/n)=\alpha$ $(0\leq\alpha\leq1)$; and then for $0<t<1$, $F_\alpha(t)=\pi^{-1}\sin\pi\alpha\int_0^t y^{\alpha-1}(1-y)^{-\alpha}dy$ if $0<\alpha<1$, $F_0(t)=1$, $F_1(t)=0$. Among others, this result applies to certain null-recurrent events ($X_n=0$ if and only if E occurs at time n) and to denumerable, irreducible and recurrent Markov chains and finite sets of states A.

The basic result is extended to Markov processes with a continuous time-parameter under regularity condition. Various applications are given, in particular to numbers of positive partial sums of independent and identically distributed summands, to sums of stable random variables and to processes with stationary independent increments.

M. Loève (Berkeley, Calif.)

4889:

✱Prouza, Ludvík. Bemerkung zur linearen Prediktion mittels eines lernenden Filters. Transactions of the first Prague conference on information theory, statistical decision functions, random processes held at Liblice near Prague from November 28 to 30, 1956, pp. 37–41. Publishing House of the Czechoslovak Academy of Sciences, Prague, 1957. 354 pp. Kčs. 34.00.

Let $\{\xi_n, -\infty<n<+\infty\}$ be a stationary stochastic process with covariance $B(\cdot)$, and let $\xi=(\xi_N, \cdots, \xi_1)$ induce the measure μ on N-space, $N\geq1$. The author's first result is that the least squares prediction equation (1) $\sum_1^N a_kB(k-l)=B(m+l)$, $m\geq0$, has a unique solution vector a whenever every $(N-1)$-dimensional hyperplane has μ-measure less than one. He then goes on to show that if $a_k(M)$ is defined as the solution of (1) with $B(k)$ replaced by the estimate (2) $M^{-1}\sum_1^M\xi_n\xi_{n+k}$, then $a_k(M)\to a_k$ almost surely if (2) $\to B(k)$ almost surely and the determinant of (1) does not vanish.

V. E. Beneš (Murray Hill, N.J.)

4890:

Il'yašenko, A. A. Des lois asymptotiques pour les chaînes de Markoff avec un nombre fini d'états. Ukrain. Mat. Ž. **10** (1958), no. 1, 23–36. (Russian. French summary)

Consider a Markov chain with a finite number of states and stationary transition. Starting from a given state let ξ_n be the number of visits to this state in the first n steps. All possible limit laws of ξ_n properly normed have been determined by Dobrushin [Izv. Akad. Nauk SSSR. Ser. Mat. **17** (1953), 291–330; MR **15**, 329] when the number of states is two. A method is indicated here which depends on the known connection between ξ_n and the successive recurrence times to the given state, and a technical device involving the inversion of Laplace-Stieltjes transforms. Some limit theorems are announced as applications of this method.

K. L. Chung (Syracuse, N.Y.)

4891:

Lehman, R. Sherman; and Weiss, George H. A study of the restricted random walk. J. Soc. Indust. Appl. Math. **6** (1958), 257–278.

As here studied, a restricted random walk is a random walk on lattice with steps of constant length such that no point on the lattice is visited more than once by the walker. The walk either continues indefinitely or the walker finally finds himself at a point where all adjacent points have been visited previously, in which case he is said to be trapped. It is first shown that on rectangular lattices in two dimensions the probability, p_n, that a walk survives n steps decreases exponentially with n, but it is remarked that the result holds for more general lattices in higher dimensions and in cases where there must be a right angle turn at each step, where probabilities are unequal for turns in different directions, and where there is an infinite barrier. From this it is further shown that, with probability one, a walker will be trapped and that the expected number of steps before being trapped is finite.

The authors then turn to the consideration of the number of n-step walks. It had already been shown that if this number is $e^{nK(n)}$, $K(n)$ approaches a constant as n tends to infinity. It is remarked that $\log 2 \leq s \leq \log 3$. Some extensive Monte Carlo experimentation using an electronic computer to study p_n, the expected value of $K(n)$, and $\langle r_n^2 \rangle$, the mean square end-to-end distance of an n-step restricted random walk, is described and discussed.

C. C. Craig (Ann Arbor, Mich.)

4892:

Kovalenko, I. M. Determining the correlation functions of some processes associated with serving problems. Dopovidi Akad. Nauk Ukraïn. RSR **1958**, 480–481. (Ukrainian. Russian and English summaries)

"The law of power utilization by machines may be interpreted as a stochastic process generated by some sequence of independent random variables. The correlation function of the process is obtained."

Author's summary

4893:

Gnedenko, B. V. On a problem of mass service. Dopovidi Akad. Nauk Ukraïn. RSR **1958**, 477–479. (Ukrainian. Russian and English summaries)

"This paper is an answer to a question which has been raised by specialists in gas and electric design. There is a great number of power consumers. The intensity of consumption at any moment of time is a random variable. Making most general assumptions, the total consumption at arbitrary moments of time $t_1, t_2, \cdots, t_s$ for any integer s is a random vector whose distribution is close to normal."

Author's summary

4894:

Karlin, Samuel; and McGregor, James. Linear growth birth and death processes. J. Math. Mech. **7** (1958), 643–662.

This is the fifth of the authors' series of papers on the integral representations for stochastic processes of the 'birth-and-death' and allied types. Its four predecessors were Trans. Amer. Math. Soc. **85** (1957), 489–546 [MR **19**, 989]; ibid. **86** (1957), 366–400 [MR **20** #1363]; Pacific J. Math. **8** (1958), 87–118 [MR **20** #3611]; and Illinois J. Math. 3 (1959), 66–81 [MR **20** #7352]. The general theme of these papers is the representation of the transition probabilities in the form

$$(*) \qquad p_{ij}(t)=\pi_j \int_0^\infty e^{-t\tau} Q_i(\tau) Q_j(\tau) d\psi(\tau),$$

where the real numbers π_j, the positive Borel measure ψ and the polynomials Q_i (orthogonal with respect to ψ) are found in terms of the 'birth rate' $\beta_i \equiv p'_{i,i+1}(0)$ and the 'death rate' $\delta_i \equiv p'_{i,i-1}(0)$. Representations like (*) for random walks were given by Kac [Amer. Math. Monthly **54** (1947), 369–391; MR **9**, 46] and for birth-and-death processes by Ledermann and Reuter [Philos. Trans. Royal Soc. London Ser. A **246** (1954), 321–369; MR **15**, 625], but the authors have attacked these problems afresh from a new starting point (a connexion with the Stieltjes moment problem) and have been rewarded by the discovery of an astonishing amount of detail. The present paper approaches most closely the earlier work of Ledermann and Reuter, for here the authors explore exhaustively the cases in which each of β_i and δ_i is linear in the 'population size', i. It turns out that for $\beta_i \equiv \lambda i + b$, $\delta_i \equiv \mu i + d$, the spectrum of ψ is discrete when $\lambda \neq \mu$ and is continuous when $\lambda = \mu$, and that the polynomials Q_i are Meixner polynomials in the first case and Laguerre polynomials in the second. In each case separate treatment is necessary for absorbing/reflecting behaviour at the lowest state, and in the first case the possibilities $\lambda > \mu$ and $\lambda < \mu$ have to be distinguished. The resulting explicit formulas for $p_{ij}(t)$ are new, although several special cases were known previously. As might be expected, the spectral point $\tau=0$ is isolated only when $\lambda \neq \mu$ (its distance from the next point in the spectrum being proportional to $|\lambda-\mu|$); when $\lambda=\mu$ the continuous spectrum covers the whole interval $[0, \infty)$. These facts elucidate the order of magnitude of $|p_{ij}(t)-p_{ij}(\infty)|$ for large t. The authors classify all the linear growth processes according to recurrence /transience, positivity/nullity, etc., and they obtain many useful results concerning first-passage-time distributions, recurrence-time distributions, 'the number of transitions before extinction', and the like. The paper concludes with a brief account of the continuous Ehrenfest model. Here the spectrum is discrete, and the associated polynomials are the Krawtchouk polynomials.

D. G. Kendall (Oxford)

4895:

McFadden, J. A. The axis-crossing intervals of random functions. IRE Trans. **IT-2** (1956), 146–150.

This paper considers the zeroes (changes of sign) of stationary ergodic processes $x(t)$ which are restricted to take on the values ± 1. Such processes occur when arbi-

trary random processes are "clipped". The autocorrelation of $x(t)$ is denoted by $r(\tau)$. The author studies the relationship between $r(\tau)$ and $P(\tau)$, the probability density of the spacing between consecutive zeroes. The following result is obtained: Under certain regularity hypotheses, $r(\tau)$ is linear in $[0, T)$ if and only if $P(\tau)=0$, $0\leq\tau<T$. The mean zero density, and approximations of $P(\tau)$ are also considered. *E. Reich* (Minneapolis, Minn.)

4896:

McFadden, J. A. The axis-crossing intervals of random functions. II. IRE Trans. **IT-4** (1958), 14–24.

The study of the quantity $P(\tau)$ defined in the paper reviewed above is continued. (It is now denoted by $P_0(\tau)$.) By applying a renewal argument an approximate solution for $P_0(\tau)$ is obtained when the intervals between successive zeroes of $x(t)$ are independent. The author considers a somewhat more general case when this independence assumption is modified. Under the assumption that the successive zero intervals form a Markov chain in the wide sense, integrals are found which yield the variance and the correlation coefficient of the lengths of the zero intervals. *E. Reich* (Minneapolis, Minn.)

4897:

Kochen, Manfred. On the commutativity of operators in stochastic models for learning. Ann. Math. Statist. **29** (1958), 930–933.

The Bush-Mosteller reinforcement operators Q_i: $p\to\alpha_i p+(1-\alpha_i)\lambda_i$ are easy to handle when they commute, but they do not commute except in trivial cases. It is asked when there exist induced operators $T_if(p)=f(Q_ip)$ which do commute, where $f(p)$ is continuous on $[0, 1]$. The result is that $T_1T_2=T_2T_1$ operating on $f(p)$ if and only if f is periodic with period $(1-\alpha_1)(1-\alpha_2)(\lambda_2-\lambda_1)$. There is a similar result for the $n>2$ case and some discussion of what this means in terms of the Q_i's.

M. L. Minsky (Cambridge, Mass.)

STATISTICS

See also 4916.

4898:

Stuart, Alan. Equally correlated variates and the multinormal integral. J. Roy. Statist. Soc. Ser. B. **20** (1958), 373–378.

The author points out that, in certain cases when the problem is to find the probability distribution or expected value of a given function of the random variables $Y_1, \cdots, Y_n$, it is more convenient to express $Y_1, \cdots, Y_n$ as linear functions of other random variables $X_1, \cdots, X_{n+1}$, and work out the problem in terms of $X_1, \cdots, X_{n+1}$. Examples are given where $Y_1, \cdots, Y_n$ are jointly normally distributed, while $X_1, \cdots, X_{n+1}$ are independent normal variables. *L. Weiss* (Ithaca, N.Y.)

4899:

Ochoa, J. Bounds for variance. Euclides, Madrid **16** (1956), 273–278. (Spanish)

This brief note indicates certain lower and upper bounds for the variance of a given discrete distribution. Though stated in different, and much more elaborate terms, the first in essence expresses the obvious fact that if the data are grouped in any way, the variance is smaller for the grouped than for the original data. A second lower bound is given by $Z^2[P(|\xi-a|>Z)+\sum_{r=1}^\infty 2^{-2r}P(2^{-r}Z<|\xi-a|\leq 2^{-r+1}Z)]$, where ξ is a random variable having the given distribution, a is the mean of the distribution, and Z is arbitrary.

An upper bound is shown to be $p_{\min}{}^{-1}\sum_r t_r$, where the data are grouped in such a way that no two observations on opposite sides of the mean fall in the same group, t_r denotes the contribution of the rth group to the (zero) average deviation from the mean for the entire distribution, and $p_{\min}$ is the smallest probability associated with a single value of the random variable in the original distribution. If the given distribution consists of the values x_i $(i=1, 2, \cdots, n)$ with probability p_i, it is further shown that the variance is less than $(a-x_1)(x_r-a)+(x_n+x_{r+1}-x_1-x_r)\tau_2-(x_nx_{r+1}-x_1x_r)s_2$, where $s_2=\sum_{i=r+1}^n p_i$, $\tau_2=\sum_{i=r+1}^n x_ip_i$, and $1\leq r\leq n$. If $r=n$, this becomes $(a-x_1)(x_n-a)$.

A method is suggested by which the bounds given can be improved. *T. N. E. Greville* (Washington, D.C.)

4900:

Weiss, Lionel. Limiting distributions of homogeneous functions of sample spacings. Ann. Math. Statist. **29** (1958), 310–312.

4901:

Steck, George P. A uniqueness property not enjoyed by the normal distribution. Ann. Math. Statist. **29** (1958), 604–606.

Let X_1 and X_2 be independent random variables each having the distribution $f(x)dx$ $(-\infty<x<\infty)$, and suppose that the quotient $U\equiv X_2/X_1$ is known to have the Cauchy distribution

$$(*)\qquad \frac{1}{\pi}\frac{du}{1+u^2}\,(-\infty<u<\infty);$$

what can we say about the density f? The author finds six possible forms for f which are distinct from the Gaussian law $f_0(x)=(2\pi)^{-\frac12}e^{-x^2/2}$; the simplest of these are

$$(**)\qquad f_1(x)=\frac{\sqrt{2}}{\pi}\,\frac{1}{1+x^4},\quad \frac{\sqrt{2}}{\pi}\,\frac{x^2}{1+x^4},$$

and so these determine statistical populations for which "Student's test with 1 degree of freedom" continues to hold. To see that this is so we have only to note that $(X_1-X_2)/(X_1+X_2)=(1-U)/(1+U)$ will enjoy the distribution (*) whenever U does so.

{From this point of view (the inversion of Student's argument) the author's problem has already been considered by J. G. Mauldon [#4884 above], who showed that the general solution is $f(x)=\pi^{-\frac12}g(x^2)$, where g satisfies the integral equation

$$(***)\qquad \int_0^\infty g(t)g(at)dt=(1+a)^{-1}\quad (a>0).$$

The equation (***) was studied by Ramanujan [see Ch. XI of G. H. Hardy, "Ramanujan", Cambridge Univ. Press, 1940; MR 3, 71] and a rigorous treatment (for $g\in L_2$) can be found in the work of F. M. C. Goodspeed [Canad. J. Math. 2 (1950), 223–237; MR 11, 661] concerning an associated Watson transform. Mauldon, in the paper cited, quoted two of the non-exponential solutions to (***) noted by Goodspeed and from them derived the two distributions (**) which yield a Cauchy law for X_1/X_2. The problem continues to attract attention; R. G. Laha [Proc. Nat. Acad. Sci. U.S.A. **44** (1958), 222–223; MR 19, 1201] has also recently rediscovered one of the

Mauldon distributions (**). In this connexion it seems worth drawing attention to a remark of Goodspeed in the paper already mentioned: If any function $g(\cdot)$ of the class L_2 satisfies (***), then so does any Watson transform of $g(\cdot)$. It should further be noted that Mauldon, in the paper cited, discussed a more general problem (his "Problem II") concerning the inversion of the argument leading to the F-test for the comparison of variances. The problem discussed here is actually the case $a=\frac{1}{2}$ of "Problem II", so that Mauldon's equation (25), with $a=\frac{1}{2}$, actually yields a general solution to (***). Mauldon obtains this by studying the L_2-solution given by Goodspeed in relation to the extra side-conditions imposed by the statistical application.}
D. G. Kendall (Oxford)

4902:

Sprott, D. A. The method of maximum likelihood applied to the Poisson binomial distribution. Biometrics **14** (1958), 97–106.

4903:

Blyth, Colin R. Note on relative efficiency of tests. Ann. Math. Statist. **29** (1958), 898–903.

It is customary to compare a test with power $\beta_n(\delta)$ against another with the same level α and power $\beta_n^*(\delta)$ by the ratio n^*/n, where n^* is defined by $\beta_n(\delta)=\beta^*_{n^*}(\delta)$. Blyth suggests that, instead, we equate the sample sizes and examine $L_1^n(\delta)=\beta_n^*(\delta)-\beta_n(\delta)$ or $L_2^n(\delta)=1-\beta_n(\delta)/\beta_n^*(\delta)$. For comparing the sign test against Student's test (one-sided normal shift alternatives), he gives the quantities $R_i^n=\operatorname{Max}_\delta L_i^n(\delta)$ for $\alpha=.05$ and $n=2(1)13$; and the R_i^∞ for six values of α, on which they heavily depend.
J. L. Hodges, Jr. (Berkeley, Calif.)

4904:

Ghurye, S. G. Note on sufficient statistics and two-stage procedures. Ann. Math. Statist. **29** (1958), 155–166.

The author investigates the method underlying Stein's two-stage procedure [same Ann. **16** (1945), 243–258; MR **7**, 213] and shows it to be applicable to the problem of testing the location parameter of an exponential distribution. The determination of the optimum two-stage procedure among a number of possible methods is also considered.
E. L. Lehmann (Berkeley, Calif.)

4905:

✱Rajski, C. The selectivity of the parametric tests. Transactions of the first Prague conference on information theory, statistical decision functions, random processes held at Liblice near Prague from November 28 to 30, 1956, pp. 33–34. Publishing House of the Czechoslovak Academy of Sciences, Prague, 1957. 354 pp. Kčs. 34.00.

4906:

Törnqvist, Leo. A method for calculating changes in regression coefficients and inverse matrices corresponding to changes in the set of available data. Skand. Aktuarietidskr. **1957**, 219–226 (1958).

The problem discussed in this paper is the adjustment in the regression coefficients resulting from change in, or addition to, the original set of variate measures. Appropriate formulas featuring the adjustment in the matrix, A, of the normal equations, and the solution of the normal equations, when a multivariate observation is added, are first presented. It is proposed then to extend the method to the determination of a whole series of regression coefficients for systems which have some different elements such as time series with different lags. Instead of deleting the oldest data and making the corresponding adjustments, the author subtracts from the old moments such a fraction of them that, after adding the term $x_i x_j$, the mean lag remains unchanged. He hence subtracts ρA from A, where $\rho=\tau/T+\tau$, T is the mean lag, τ is the time interval between successive observations, and then develops the adjustment formulas. He also develops more general adjustment formulas most of which are related, as he indicates, to work which has been previously published. The material in section 3, featuring the "discounting factor" $1-\rho$, appears to be completely new.
P. S. Dwyer (Ann Arbor, Mich.)

4907:

Doornbos, R.; and Prins, H. J. On slippage tests. III. Two distributionfree slippage tests and two tables. Nederl. Akad. Wetensch. Proc. Ser. A **61**=Indag. Math. **20** (1958), 438–447.

Analogues to the inequalities of Parts I and II [same Proc. **61** (1958), 38–55; MR **20** #4339] are found for the Friedman-Kendall and Wilcoxon-Mann-Whitney-Kruskal-Wallis rankings. Significance levels are supplied for the decision procedures with Poisson populations (part I, sect. 2) and the Friedman-Kendall rankings.
I. R. Savage (Minneapolis, Minn.)

4908:

Claringbold, P. J. Multivariate quantal analysis. J. Roy. Statist. Soc. Ser. B. **20** (1958), 398–405.

For analysis of covariance problems where the possible values of each observation are 0, 1, the author discusses the estimation of the linear transformation that yields the canonical variates. Numerical examples are given.
L. Weiss (Ithaca, N.Y.)

4909:

Washio, Yasutoshi. A note on the point estimation in N-way layout. Mem. Fac. Sci. Kyusyu Univ. Ser. A. Math. **11** (1957), 157–165.

In a 2-way layout with at least two replications in each cell with the usual fixed effects model in the analysis of variance, it is known that the unbiased estimates of the main and interaction effects and their sums of squares are best in the sense of minimum risk. It is observed that the proof goes through for an n-way layout and that it is not required that the number of replications be constant in each cell so long as there is a minimum of 2. In the case of a 1-way layout with random effects the like result is established, but it is observed that the method used does not extend to layouts of more than one dimension with random effects. The author points out that, in a recent paper by Graybill and Wortham [J. Amer. Statist. Assoc. **51** (1956), 266–268; MR **18**, 78] dealing with this last situation, statistics stated to be sufficient are not so in fact, which invalidates their proof.
C. C. Craig (Ann Arbor, Mich.)

4910:

Kiefer, J. On the nonrandomized optimality and randomized nonoptimality of symmetrical designs. Ann. Math. Statist. **29** (1958), 675–699.

The author discusses various criteria of optimality of experimental designs. Extending rather special results due to A. Wald [same Ann. **14** (1943), 134–140; MR **5**, 129] and S. Ehrenfeld [ibid. **26** (1955), 247–255; MR **17**, 56] he shows that many commonly employed symmetrical designs (such as balanced incomplete block designs, Latin squares, Youden squares, etc.) possess optimum properties among the class of non-randomized designs. He then

proceeds to show that, surprisingly, all these optimum results fail to hold if randomized designs are considered. The underlying populations are assumed, as customary, to be normal but the author's method applies more generally (in the non-optimality results the point is, of course, not that optimality fails in many models but that it fails in the simplest, classical, normal model). The paper concludes with some remarks and open problems.

A. Dvoretzky (Jerusalem)

4911a:

Kshirsagar, A. M. A note on incomplete block designs. Ann. Math. Statist. **29** (1958), 907–910.

4911b:

Mote, V. L. On a minimax property of a balanced incomplete block design. Ann. Math. Statist. **29** (1958), 910–914.

In the setting where (A) balanced incomplete block designs (BIBD's) and (B) Youden squares (YS's) are customarily used, let C be the class of all designs for which each variety occurs at most once in any block in case (A), and at most once in any row or any column in case (B). Let θ denote the u-vector of treatment effects. In the special case of case (B) where a Latin square (LS) can be used, Wald [same Ann. **14** (1943), 134–140; MR **5**, 129] showed that the LS design minimizes (1) the generalized variance of the best linear estimators (b.l.e.'s) of any $u-1$ linearly independent contrasts $t_i'\theta$ (with analogous results for higher LS's), and Ehrenfeld [ibid. **26** (1955), 247–255; MR **17**, 56] showed that the LS design minimizes (2) the maximum, over all t with $t't=1$, of the variance of the b.l.e. of the contrast $t'\theta$. Mote shows, in case (A), that the BIBD minimizes (2) among all designs in C. Kshirsagar shows that, among all designs in C, the BIBD in case (A) and the YS in case (B) minimize (1) and also minimize (3) the average variance of any $u-1$ orthogonal contrasts $t_i'\theta$. Part of a paper of the reviewer which appeared simultaneously [#4910 above] proves, without the restriction to designs in C, that the appropriate designs in cases (A) and (B) (and generalizations of them in the case where block, row, or column size is $>u$) minimize (1), (2), and (3) and have certain other optimum properties, with analogous results in other symmetrical design settings.

J. Kiefer (Oxford)

4912:

Jiřina, Miloslav; and Nedoma, Jiří. Minimax solution of sampling inspection plan. Apl. Mat. **1** (1956), 296–314. (Czech. Russian and English summaries)

For given lot-size N, process average fraction defective $\bar{p}$ and conditions $H(p_t, c, n, N)=\varepsilon$ or

$$\max_{0\leq p\leq 1} H(p, c, n, N)(1-n/N)p=p_L,$$

where $H(p, c, n, N)$ is the probability of accepting the lot of size N with lot fraction defective p by the acceptance number c and the sample size n, a sampling inspection plan with complete inspection of rejected lots, minimizing the expression $\max_{F\in\mathcal{F}(\bar{p})} I(c, n, N, F)$, is presented. The value $I(c, n, N, F)=N-(N-n)\int_0^1 H(p, c, n, N)dF(p)$ corresponds to the average number of pieces inspected per lot if the fraction defective p is distributed according to the distribution function $F(p)$. $\mathcal{F}(\bar{p})$ denotes the system of all distribution functions satisfying $\int_0^1 dF(p)=1$ and $\int_0^1 p\,dF(p)=\bar{p}$. Charts and tables are given to find the sample size n and the acceptance number c for the plan described.

J. Janko (Prague)

NUMERICAL METHODS

See also 4567, 4715, 4891, 5077, 5102.

4913:

Bauer, W. F. The Monte Carlo method. J. Soc. Indust. Appl. Math. **6** (1958), 438–451.

Expository paper, with some new results on elliptic partial differential equations.

4914:

Bizley, M. T. L. A measure of smoothness and some remarks on a new principle of graduation. J. Inst. Actuar. **84** (1958), 125–165.

The author criticizes the ordinary finite difference way of smoothing curves. This way depends on the actual situation of the x and y axes. By minimizing, instead, the differential coefficient of the coefficient of curvature, the graduation becomes independent of the system of coordinates. No immediate results are given, but the paper is intended to uncover a new field for graduating research.

P. Johansen (Copenhagen)

4915:

Duquenne, René. Sur le calcul rhéographique et numérique des fonctions harmoniques définies dans tout l'espace. C. R. Acad. Sci. Paris **247** (1958), 263–266.

"Parmi différentes méthodes destinées à généraliser à l'espace l'emploi de l'inversion pour représenter sur deux domaines limités une fonction harmonique définie dans tout le plan, nous utilisons l'inversion de Kelvin. Le changement de fonction introduit une discontinuité dont la représentation analogique est très simple dans un cas particulier."

Résumé de l'auteur

4916:

Akaike, Hirotugu. On a computation method for eigenvalue problems and its application to statistical analysis. Ann. Inst. Statist. Math. **10** (1958), 1–20.

This paper deals with the solution of $Ax=\lambda Bx$ where B is an N by N positive definite real symmetric matrix, A is an N by N real symmetric matrix and x is an N-dimensional column vector, by an iterative method which is appropriate for large N. This equation is a basic equation in the theory of socalled canonical correlation. The author relates it, in a final section, to other statistical concepts.

The problem may appear in different forms. Premultiplication by B^{-1} gives $(B^{-1}A-\lambda I)x=0$, a conventional form used in statistics though $B^{-1}A$ is not in general symmetric. The author proposes to obtain a solution which does not demand the calculation of B^{-1}. Now premultiplication by x' leads to $x'Ax/x'Bx=\lambda$ and we define $x'Ax/x'Bx=\mu(x)$. If only the largest characteristic root is desired, it is only necessary to find the maximum value of $\mu(x)$.

The iterative method features the construction of a set of vectors $\xi_1(x)$, $\xi_2(x)$, $\cdots$, $\xi_k(x)$, where the $\xi_i(x)$ are continuous functions of x and where x and the ξ_i are linearly independent. Special attention is given to the case in which $k=1$ with $\xi_1=\xi=Ax-\mu(x)Bx$. When $\lambda_1>\lambda_2$ and $x^{(n+1)}=x^{(n)}+\lambda_{(n)}\xi^{(n)}$, with $\lambda_{(n)}$ a function of $x^{(n)}$ and $\xi^{(n)}$, the process converges to x_1. For the general ξ, and for more general computation schemes, no proof is available that the process converges to x_1, though the author has provided a 5-step check for comparing the solution with λ_1.

The use of $\xi=Ax-m(x)Bx$ is also illustrated, where $m(x)=A'x'x'A/B'x'xB$, and there is discussion and illustration of an acceleration scheme. There is some discussion of the general process based on the set x, ξ_1, ξ_2, $\cdots$, ξ_N. *P. S. Dwyer* (Ann Arbor, Mich.)

4917:

Stancu, D. D. Sur certaines formules générales d'intégration numérique. Acad. R. P. Romîne. Stud. Cerc. Mat. 9 (1958), 209–216. (Romanian. Russian and French summaries)

Let $f(x)$ be a function defined in an interval (a, b). With the interpolation formula of Lagrange-Hermite $H_{n-1}(x;f)=\sum_{i=1}^{s}\sum_{k=0}^{r_i-1} l_{i,k}(x)f^{(k)}(x_i)$, where the $l_{i,k}$ are polynomials of interpolation of degree at most $n-1$, write $f(x)=H_{n-1}(x;f)+R(x;f)$, where

$$R(x;f)=u(x)[x, x_1, \cdots, x_1, \cdots, x_s, \cdots, x_s; f],$$

Here each x_i occurs r_i times within the bracket, and $u(x)$ is a polynomial of degree n with real roots, $u(x)=\prod_{i=1}^{s}(x-x_i)^{r_i}$, $n=r_1+r_2+\cdots+r_s$. From the above the author develops formulas for the calculation of the coefficients of the general formula of numerical integration $\int_a^b p(x)f(x)dx=\sum_{i=1}^{s}\sum_{k=0}^{r_i-1} A_{i,k}f^{(k)}(x_i)+\rho[f]$, where $p(x)$ is a given function, integrable in (a, b). Following this same formula of interpolation, he develops the polynomial $h(x)$ of degree smaller than n, and obtains expressions for the coefficients $B_{i,k}$ of the development $R(x)=h(x)/u(x)=\sum_{i=1}^{s}\sum_{k=0}^{r_i-1} B_{i,k}/(x-x_i)^{k+1}$. Comparison of formulas for the $A_{i,k}$ and $B_{i,k}$ yield relations given by Tchakaloff [C. R. Acad. Bulgare Sci. Math. Nat. 1 (1948), 2–3, 9–12; Dokl. Akad. Nauk SSSR 68 (1949), 233–236; MR 10, 743; 11, 236]. Finally, the author gives an extension of the preceding results to the case of two variables.

E. Frank (Chicago, Ill.)

4918:

Selmer, Ernst S. Numerical integration by non-equidistant ordinates. Nordisk Mat. Tidskr. 6 (1958), 97–108, 136.

In this paper the author discusses various numerical integration formulas in which the evaluation points do not form an arithmetic progression. Error terms are given by consideration of Taylor series and by substitution.

P. C. Hammer (Madison, Wis.)

4919:

Černyšenko, È. A. On a method of approximate solution of Cauchy's problem for ordinary differential equations. Ukrain. Mat. Z. 10 (1958), no. 1, 89–100. (Russian)

The author gives an iterative method for solving the initial value problem for an nth order ordinary differential equation which is similar to Picard's method of successive approximations. He proves that the method converges geometrically in a sufficiently small interval. In general, the method involves solving a system of non-linear equations for each iteration step, although if the differential equation is linear, the system of equations is also linear. An example is given showing that the given method is more accurate than Picard's but much more labour is required per iteration, including the solution of a transcendental equation. *P. Rabinowitz* (Rehovoth)

4920:

Gerberich, C. L.; and Sangren, W. C. Codes for the classical membrane problem. J. Assoc. Comput. Mach. 4 (1957), 477–486.

The problem of concentration of fissionable material and neutron denisty distribution for a bare reactor, in elementary nuclear reactor theory, corresponds roughly to the fundamental eigenvalue and eigenfunction of the classical membrane problem. Mathematically, the problem is $\nabla^2\phi+\lambda\phi=0$ in a region R, $\phi=0$ on the boundary of R. The authors set this up as a difference equation in two dimensions, and hence consider a matrix eigenvalue problem. The numerical analysis utilizes a modification of the "power" method (or method of Stodola and Vianello) suggested by Flanders and Shortley [J. Appl. Phys. 21 (1950), 1326–1332; MR 12, 640]. This numerical problem was then coded and run on the ORACLE at Oak Ridge National Laboratory, using a large number of geometries of a general class including rectangular, L-shaped, cross-shaped, trapezoidal and square-doughnut-shaped regions, the boundary geometries being generated by a special sub-routine. Other sub-routines were used to study the approximate eigenvalues as a function of mesh-size [cf. Forsythe, Pacific J. of Math. 4 (1954), 467–480; MR 16, 179], to evaluate the Rayleigh quotient, and to locate equi-potential curves or isopleths. Brief tables of results are included. *R. B. Davis* (Syracuse, N.Y.)

4921:

Keller, H. B.; and Wendroff, B. On the formulation and analysis of numerical methods for time dependent transport equations. Comm. Pure Appl. Math. 10 (1957), 567–582.

The transport equation frequently encountered in radiation and neutron transport is of the form

$$(1)\qquad [V^{-1}(\partial/\partial t)+\mathbf{w}\cdot\nabla+\sigma(t, \mathbf{r}, v)]\Phi=S(\Phi; t, \mathbf{r}, \mathbf{w}; v).$$

The unknown for which we must solve is the (neutron) density $\Phi(t, \mathbf{r}, \mathbf{w}, v)$ as a function of position in space ($\mathbf{r}$), of the incident beam (of neutrons, say) having speed v and velocity direction $\mathbf{w}$, and of the time t. In equation (1), σ denotes the total cross section for scattering and absorption, while S represents creation of particles by emission, fission, etc.

Following some ideas of B. G. Carlson [Los Alamos Report **LA-1891**, 1955], and allowing two possible geometries (plane and spherically symmetric), the authors proceed as follows. Equation (1) is assumed in the form

$$(2)\qquad [V^{-1}(\partial/\partial t)+\mu(\partial/\partial r)+r^{-1}(1-\mu^2)(\partial/\partial\mu)+\sigma(r)]\Phi(t, r, \mu) =S(\Phi; t, r, \mu).$$

Introducing finite differences for μ, one obtains a hyperbolic system (3) of first-order linear partial differential equations, which can be taken in normal form, and integrated along the characteristics in the r, t plane. In order to carry out this integration numerically, the authors place a mesh on the r, t plane and get a system (4) of difference equations. There are now three problems: (i) the system (4) requires solution by iteration, so one must prove that this iteration converges; (ii) it is necessary to prove that the solution of the difference equations (4) converges to the solution of the hyperbolic partial differential equations (3) as the r, t mesh is refined; finally, (iii) it is necessary to prove that the solution of (3) converges to the solution of (2) as the μ mesh is refined. The authors prove (i) and (ii) in the present paper. In a separate paper, to appear later, they prove (iii). If neither scattering nor fission is present a direct solution of (4) is possible and step (i) can be omitted.

R. B. Davis (Syracuse, N.Y.)

4922:

Lučka, A. Yu. A sufficient condition for the convergence of the procedure for averaging functional corrections. Dokl. Akad. Nauk SSSR **122** (1958), 179–182. (Russian)

Consider the Fredholm integral equation of the second kind

$$(1)\quad y(x)=\varphi(x)+\lambda\int_a^b K(x,\xi)y(\xi)d\xi\ (0<|\lambda|<\infty),$$

where $\varphi(x)$ and $K(x,\xi)$ are real-valued and belong to $L_2(a,b)$ and let (1) have the unique solution $Y(x)$ for a specified value of the parameter λ. Now define the sequence $\{y_n(x)\}$ by the relations

$$(2)\quad y_n(x)=\varphi(x)+\lambda\int_a^b K(x,\xi)[y_{n-1}(\xi)+\alpha_n]d\xi,\ y_0(x)\equiv 0$$

$$\alpha_n=\frac{1}{h}\int_a^b \delta_n(x)dx\ (h=b-a>0),$$

$$\delta_n(x)=y_n(x)-y_{n-1}(x)\ (n=1,2,3,\cdots),$$

which in turn imply the relations

$$(3)\quad \delta_n(x)=\lambda\int_a^b K(x,\xi)[\delta_{n-1}(\xi)-\alpha_{n-1}]d\xi+\lambda\alpha_n\int_a^b K(x,\xi)d\xi \quad (n=2,3,\cdots),$$

$$\alpha_n=\frac{\lambda}{D(\lambda)}\int_a^b\int_a^b K(x,\xi)[\delta_{n-1}(\xi)-\alpha_{n-1}]d\xi dx \quad (n=2,3,\cdots),$$

where

$$(4)\quad \alpha_1=\frac{1}{D(\lambda)}\int_a^b\varphi(x)dx,\ D(\lambda)=h-\lambda\int_a^b\int_a^b K(x,\xi)d\xi dx.$$

It is shown that this defines a convergent process, with $y_n(x)\to Y(x)$, if $L^2<1$, where

$$L^2=\lambda^2(B^2-hM^2)\left\{1+\frac{|\lambda|h^{3/2}}{|D(\lambda)|}[M^2-hK^2]^{\frac12}\right\}^2,$$

$$B^2=\int_a^b\int_a^b K^2(x,\xi)d\xi dx,\ M^2=\frac{1}{h^2}\int_a^b\left(\int_a^b K(x,\xi)d\xi\right)^2dx,$$

$$K=\frac{1}{h^2}\int_a^b\int_a^b K(x,\xi)d\xi dx.$$

An example is given for which $L^2<1$ but for which $\lambda^2B^2>1$. It will be recalled that $\lambda^2B^2<1$ guarantees the convergence of the usual "successive approximations" procedure for solving (1). Thus the process here defined appears to have considerable merit.

J. F. Heyda (Cincinnati, Ohio)

4923:

Golubeva, K. I. Application of a trilinear correspondence to certain questions of nomography. Moskov. Oblast. Pedagog. Inst. Uč. Zap. **57** (1957), 207–230. (Russian)

COMPUTING MACHINES

See also 4492, 4493, 4494, 4920, 5111.

4924:

Tocher, K. D. Techniques of multiplication and division for automatic binary computers. Quart. J. Mech. Appl. Math. **11** (1958), 364–384.

Multiplication in a binary computer is conventionally performed by adding the multiplicand to the shifting partial product whenever a "one" is encountered in the multiplier. It is possible, however, to take advantage of the subtraction facility to reduce the total number of addition-subtraction operations. To achieve this reduction the multiplier N is represented in a recoded form

$$N=\sum_r(-)^{S_r}d_r2^r\ (S_r,d_r=0,1;\ r=0,1,\cdots),$$

in which the number of non-zero d_r is minimized. Recursion formulas are given which yield this minimization. Under certain circumstances these formulas may be utilized to effect a saving of multiplication time in parallel machines and a saving of time and equipment in serial machines.

Similar schemes are studied for the case of division.

D. E. Muller (Urbana, Ill.)

4925:

Sauer, R. Über die Münchner Rechenanlage "PERM" und die Entwicklung der numerischen Mathematik. Univ. e Politec. Torino. Rend. Sem. Mat. **16** (1956–57), 39–54.

A description of the automatic digital computer at the Munich Institute of Technology. It is an 8192 word magnetic drum machine with an access time of 2 msec.

4926:

van der Sluis, A. Curves reproduced by the harmonic analyzer. Simon Stevin **32** (1958), 39–41. (2 plates) (Dutch)

This paper gives a brief description of the Mader-Ott harmonic analyser [cf. Meyer zur Capellen "Mathematische Instrumente" 2d ed., Akademischer Verlag, Leipzig, 1944; MR 9, 160] and then describes the curves traversed by the sine and cosine points on the intermediate wheel of this analyser for various simple functions.

J. G. L. Michel (Teddington)

4927:

Medgyessy, Pál. A mechanical functional synthesizer. Magyar Tud. Akad. Mat. Kutató Int. Közl. **2** (1957), 33–42. (Hungarian and Russian summaries)

The author describes a mechanical analog device for computing partial sums, $\sum_{k=0}^N a_kf_k(x)$, on a finite set of points, x_i, $i=1,\cdots,m$. With some tinkering, the gadget can be altered to accommodate any given set of functions, $f_k(x)$.

E. K. Blum (Los Angeles, Calif.)

MECHANICS OF PARTICLES AND SYSTEMS

4928:

Kel'zon, A. S. On the motion of a point on a pursuit curve. Bul. Inst. Politehn. Iași (N.S.) **3** (1957), 43–48. (Russian. English and Romanian summaries)

The author derives equation of motion of a point moving on a pursuit curve (i.e., a curve along which the velocity vector is continuously directed toward the moving target) for various ratios of the velocities in question. Certain results obtained here were obtained already in the past (by Bouquer, Cailler, Ficklin, etc.). An inverse problem is also treated, where the velocity vector is directed from the target along the line connecting the moving point and the target. The approach used is very elementary.

M. Z. v. Krzywoblocki (Urbana, Ill.)

4929:

Woinaroschi, R.; and Romalo, D. On the instantaneous distribution of the accelerations in the kinematics of a rigid solid. Lucrarile Inst. Petrol Gaze Bucureşti **4** (1958), 355–361. (Romanian. Russian and English summaries)

A simple geometrical property of the acceleration field is deduced in the case of a rigid solid and it is shown how this property may be used for determining the accelerations of a mechanism. *Author's summary*

4930:

Lur'e, A. I. Remarks on analytical mechanics. Prikl. Mat. Meh. **21** (1957), 759–768. (Russian)

L'auteur introduit des simplifications et des généralisations dans quelques problèmes classiques de la mécanique rationnelle. 1) Une généralisation dans le cas de la fonction dissipative où les forces de la résistance sont proportionnelles à une puissance m de la vitesse ($m=1$, Cas de Rayleigh). 2) Une expression simplifiée de la formule de l'énergie cinétique des accélérations [voir P. Appell, Traité de mécanique rationnelle, t. II, Gauthier-Villars, Paris] en introduisant les symboles de Christoffel. 3) Une étude détaillée d'un problème de Darboux [Darboux, Leçons sur la théorie générale des surfaces, t. I, Gauthier-Villars, Paris, 1941]. Il s'agit de trouver les paramètres qui définissent la position d'un corps solide, qui tourne autour d'un point fixe dont le vecteur de la rotation angulaire est donné d'avance. En utilisant les paramètres de Cayley-Klein [Klein et Sommerfeld, Über die Theorie des Kreisels, Teubner, Leipzig, 1910] le problème se réduit à une équation de Riccati. L'auteur étudie le cas où l'intégration se réduit à des quadratures.

M. Kiveliovitch (Paris)

4931:

Valentine, F. A. The motion of a flexible inelastic tube constrained to move on a rough convex curve. Amer. Math. Monthly **65** (1958), 179–184.

A plane closed differentiable convex curve is materialized as a rough wire with a constant coefficient of friction. A hollow slender flexible string is strung on the wire; this tube has uniform density. The only force acting on it is the constraint. The motion of the tube has some invariant properties (cf. the motion of a particle along a wire studied by the author before [same Monthly **63** (1956), 16–20; MR **17**, 910]). We cite the theorem: The initial conditions being given, the speed with which the tube returns to its initial position is independent of the curve and of the length of the tube. *O. Bottema* (Delft)

4932:

Magnus, K. On the stability of a heavy symmetrical gyroscope on gimbals. J. Appl. Math. Mech. **22** (1958), 237–243 (173–178 Prikl. Mat. Meh.).

The author considers the stability of a heavy symmetrical gyroscope on gimbals under the assumption that the bearings are devoid of friction and that the gimbals can have finite moments of inertia. The position of interest is that in which the planes of the gimbals are coincident and vertical and the axis of the rotor is also vertical.

After some changes of variables, the author constructs a Liapunov function from which he is able to deduce stability conditions involving the moments of inertia, the mass of the rotor, the component of the rotor's angular velocity along its axis, and the position of the center of mass of the system. It is concluded that: (1) a standing gyroscope may lose stability even without an initial push on the gimbal ring in the direction of the rotor's spin; (2) a hanging gyroscope could lose stability if it received a push of sufficient magnitude either in the direction of or opposite to its spin. *H. M. Trent* (Washington, D.C.)

4933:

✱Okhotsimskii, D. E.; and Eneev, T. M. Certain variational problems associated with the launching of an artificial earth satellite. The Russian literature of satellites, I, pp. 1–44. Translated from Uspehi Fiz. Nauk **63** (1957), no. 1a. International Physical Index, Inc., New York 1958. vi+181 pp. (1 plate) $10.00.

The problem considered is the proper utilization of thrust to raise a rocket to given altitude with maximum terminal velocity (assumed horizontal). First a number of simplifying assumptions are made, leading to the differential equations $x'=u$, $y'=w$, $u'=p\cos\phi$, $w'=p\sin\phi-g$, where x and y are horizontal and vertical coordinates and $p\cos\phi$, $p\sin\phi$ are the components of acceleration due to the rocket motor. One wishes to choose $p=p(t)$ and $\phi=\phi(t)$ so as to maximize $u(t)$, subject to boundary conditions at $t=0$ and $t=T$. The variational problem is treated with the aid of Lagrange multipliers (the details could not all be verified by the reviewer) and conclusions are reached as to optimum choice of ϕ and p; for example, $\tan\phi$ should be a linear function of t. The problem is then modified in that $p(t)$ is assumed given; a detailed study is made of the case of a multiple stage rocket and graphs of attainable altitudes and terminal velocities are given. In a final section the procedures are generalized to take into account the varying gravitational field and the rotation of the earth. *W. Kaplan* (Ann Arbor, Mich.)

4934:

Egorov, V. A. On the solution of a degenerate variational problem and the optimum climb of a cosmic [space] rocket. J. Appl. Math. Mech. **22** (1958), 20–36 (16–26 Prikl. Mat. Meh.).

In the plane of the variables x_2, x_3, let there be given a simply connected region σ_1 with a closed boundary $\sigma_1{}^0$. The variational problem considered is that of determining in σ_1 a piecewise smooth Jordan arc γ_1: $x_2=x_2(\tau)$, $x_3=x_3(\tau)$, $0\leqq\tau\leqq1$, with prescribed end points on $\sigma_1{}^0$, such that the value $x_1(1)$ is a maximum for the function $x_1(\tau)$ determined by the initial value $x_1(0)$ and along γ_1 by the equation

$$X_1(x_1,x_2,x_3)dx_1+X_2(x_1,x_2,x_3)dx_2+X_3(x_1,x_2,x_3)dx_3=0,$$

where the X_i are of class C' and $X_1\neq0$, and the tangents to the space curve $\gamma: x_i=x_i(\tau)$, $0\leqq\tau\leqq1$, $i=1,2,3$, belong to a prescribed pencil of admissible directions.

It is shown that several problems reduce to the general one formulated above: The problem of the optimal expenditure of fuel in rectilinear motion of rockets reaching maximum altitude with given fuel consumption, or reaching a given altitude with least expenditure of fuel, or reaching maximum speed at prescribed altitude; the problem of the determination of the optimal trajectory in a vertical plane in order to minimize expenditure of fuel in reaching prescribed terminal values of altitude and speed, or in order to maximize speed at prescribed altitude in a flight of prescribed duration, or in order to maximize flight altitude when flight time and final speed are prescribed; the problem of the motion of a rocket with zero angle of attack on a rigid, smooth track in which the shape of the track is to be found.

It is shown that the problem is a degenerate one; the variation is analyzed, and the general solution is given.

E. F. Beckenbach (Los Angeles, Calif.)

STATISTICAL THERMODYNAMICS AND MECHANICS

See also 5059, 5079.

4935:

★Carleman, T. Problèmes mathématiques dans la théorie cinétique des gaz. Publ. Sci. Inst. Mittag-Leffler. 2. Almqvist & Wiksells Boktryckeri Ab, Uppsala, 1957. 112 pp.

This monograph, dealing with the mathematical aspects of the Boltzmann transport equation, contains a manuscript, left unfinished by the author on his death in 1949 and edited by L. Carleson and O. Frostman, who also added some theorems.

The first part of the monograph gives a general mathematical discussion of the Boltzmann equation for the case where the distribution function depends on the velocities only. No discussion is given of the derivation of the Boltzmann equation which involves the molecular chaos assumption. In this part the author proves Boltzmann's H-theorem (which in his case is strictly valid through the use of the transport equation derived using the molecular chaos assumption), and an existence and uniqueness theorem for the solution of the Boltzmann equation. It is also shown that any solution tends asymptotically to the Maxwell distribution.

In the second part solutions of the Boltzmann equation are considered which are slight departures from the equilibrium solution (the case of elastic spheres and of no external forces is the only case discussed), and some special cases of the Boltzmann equation are also discussed.

The main discussion in this book is of the mathematical properties of the solutions and of the kernel of the integro-differential equation. *D. ter Haar* (Oxford)

4936:

Yamamoto, Misazo. The visco-elastic properties of network structure. III. Normal stress effect (Weissenberg effect). J. Phys. Soc. Japan **13** (1958), 1200–1211.

The author continues the work of two previous papers [same J. **11** (1956), 413–421; **12** (1957), 1148–1158; MR **19**, 1113] in which he develops a statistical theory for the flow of high polymer solutions. This theory is based on the concept of a weakly coupled rubberlike network model and the stress deformation relations evolved include terms which take into account the breakage and re-formation of network junctions. The author introduces an unsymmetrical deformation tensor, however, and it is not clear whether his stress deformation relations satisfy the necessary conditions for invariance under rigid body rotations. The theory is employed to examine the normal stress effect (Weissenberg effect) arising in a fluid in motion between rotating coaxal cylinders, between parallel plates and between coaxial cones. A discussion is given of the relation of the theory to experimental results. *J. E. Adkins* (Nottingham)

4937:

Block, B. Generalized transport theory. Ann. Physics **6** (1959), 37–49.

The author derives a general formula for the Hall coefficient of a system of charges in small electric and magnetic fields. The derivation is carried out classically first, then quantum-mechanically. The classical derivation however, is presented so as to correspond very closely to the quantum case. The investigation is restricted to the so-called weak coupling case of transport theory, i.e., the case where the interactions responsible for irreversible phenomena are small. The treatment of this weak coupling approximation, as well as the introduction for classical systems of a Green function in the momentum variables only, are points which in the reviewer's opinion would require further clarification and justification. *L. Van Hove* (Utrecht)

4938:

Hashitsume, Natsuki. A statistical theory of linear dissipative systems. II. Progr. Theoret. Phys. **15** (1956), 369–413.

[For part I of this series, see Prog. Theoret. Phys. **8** (1952), 461–478; MR **14**, 1048.] This second part of a series of papers on a statistical theory of linear dissipative systems contains a further investigation of the theory and a number of applications. The equivalence is proved of Onsager and Machlup's method of calculating the probability of a given succession of nonequilibrium states of a spontaneously fluctuating thermodynamic system and Wang and Uhlenbeck's theory of Brownian motion based on Rice's treatment of the Gaussian process. The thermodynamical Lagrangian, i.e., the variation function yielding Onsager's principle of least dissipation is investigated in detail to show that the value of the function becomes zero when the system obeys the phenomenological linear relations given by the thermodynamics of irreversible processes, while it increases otherwise.

The application of the theory of generalized Brownian motion to Johnson noise is made by regarding the electronic distribution function in metals as the random variables obeying the generalized Langevin equation. The correlation function of the electronic distribution function is derived in an elegant way, which yields the expression for the correlation function of the electric currents given by Bakker and Heller. The application of the fluctuating distribution function method to the theory of the shape of collision broadened spectral lines is also discussed, assuming harmonic oscillators immersed in a heat reservoir. It is shown that the method yields the same results as the strong collision treatment by Gross. *H. Mori* (Providence, R.I.)

4939:

Tauber, G. E. A generalized variational principle for transport phenomena. Canad. J. Phys. **36** (1958), 1308–1318.

The general procedure for obtaining a variational principle yielding the Boltzmann equation breaks down for electron conduction in the presence of a magnetic field [J. M. Ziman, Canad. J. Phys. **34** (1956), 1256–1273; MR **18**, 611]. A variational principle for this case was obtained by Moliner and Simons [Proc. Cambridge Philos. Soc. **53** (1957), 848–855; MR **19**, 1102]. The present paper contains a further investigation of the new variational principle, which takes into account the deviation of the phonon distribution function from equilibrium and is valid for an arbitrary direction of the electric field and polarization of the lattice vibrations. Use of the Ritz method is made to solve the variational principle and formulate the transport coefficients in terms of infinite determinants. *H. Mori* (Providence, R.I.)

4940a:

Kubo, Ryogo. Statistical-mechanical theory of irreversible processes. I. General theory and simple applications to magnetic and conduction problems. J. Phys. Soc. Japan **12** (1957), 570–586.

4940b:

Kubo, Ryogo; Yokota, Mario; and Nakajima, Sadao. Statistical-mechanical theory of irreversible processes. II. Response to thermal disturbance. J. Phys. Soc. Japan 12 (1957), 1203–1211.

There is a relation between the electrical conductivity and the correlation function of fluctuating electric currents, known as the Nyquist theorem. This theorem is extensively investigated by the author in the first paper, in order to formulate a rigorous method of calculating the electrical conductivity from molecular dynamics. In the second paper a method is presented for obtaining similar expressions for the general transport coefficients, including the thermal conductivity, in terms of the correlation functions of the appropriate dynamical quantities. The method is a straightforward extension of Onsager's derivation of the reciprocity theorem in the thermodynamics of irreversible processes.

H. Mori (Providence, R.I.)

4941:

van Kampen, N. G. Thermal fluctuations in a nonlinear system. Phys. Rev. (2) 110 (1958), 319–323.

The author calculates the current fluctuations generated by a voltage-dependent resistance $R(V)$ in contact with a heat bath of temperature T and in series with a condenser C. The starting-point is the Fokker-Planck equation $\partial P/\partial t=(\partial^2/\partial q^2)[\xi(q)P]+(\partial/\partial q)[\eta(q)P]$, where $P(q, t)dq$ is the probability that at time t the charge on C lies between q and $q+dq$. Then, by applying the principle of detailed balancing in a way described elsewhere] [van Kampen, Physica 23 (1957), 707–719; MR 20 #7408], $\eta(q)$ is eliminated from the Fokker-Planck equation to give $\partial P/\partial t=(\partial/\partial q)[\xi G(\partial/\partial q)P/G]$, where G is the equilibrium distribution

$$G(q)=(2\pi kTC)^{-\frac{1}{2}}\exp(-q^2/2kTC).$$

This equation is then solved by perturbation theory (up to the second order), where ξ is written as the sum of $\xi(0)$=const. and a small perturbation $\xi^{(1)}(q)$. The results are used to calculate first the autocorrelation function of $q(t)$ and then the spectral density of $i(t)=dq(t)/dt$. A key role is played by the identification $\xi=kT/R$, which the author justifies for sufficiently large C. Explicit calculations are given for the cases $R(V)=R_0+R_1V$ and $R(V)=R_0+R_2V^2$.

R. A. Silverman (New York, N.Y.)

4942:

Magalinskiĭ, V. B.; and Terletskiĭ, Ia. P. Calculation of the coordinate probabilities by Gibbs method. Soviet Physics. JETP 34(7) (1958), 501–504 (729–734 Ž. Eksper. Teoret. Fiz).

Using canonical ensembles, expressions are derived both for the probability distribution of a generalized coordinate q and for the function defining the correlation between a value q_1 at t_1 and a value q_2 at t_2 in terms of the response of q to an imposed generalized force along the q-direction.

D. ter Haar (Oxford)

4943:

Hori, Jun-ichi. On a relation between fluctuation-dissipation theorem and irreversible thermodynamics. J. Phys. Soc. Japan 11 (1956), 1220–1227.

The fluctuations about equilibrium of a group of thermo-dynamic variables obeying the Onsager phenomenological relations are treated by introducing random forces with 'white' correlation spectra. When, in order to consider the fluctuations of a particular variable, all others are eliminated, the resulting equation corresponds to an impedance subjected to a random force whose correlation spectrum is equal to the real part of the impedance function, in accordance with the fluctuation dissipation theorem [H. Takahashi, J. Phys. Soc. Japan 7 (1952), 439–446; H. B. Callen and T. A. Welton, Phys. Rev. (2) 83 (1951), 34–40; MR 13, 477; H. B. Callen and R. F. Greene, ibid. 86 (1952), 702–710; MR 14, 230]. The discussion is extended to include the case of velocity variables.

S. Prager (Brussels)

ELASTICITY, PLASTICITY

See also 4920, 4936.

4944:

***Aржаных, И. С. Интегральные уравнения основных задач теории поля и теории упругости. [Aržanyh, I. S. Integral equations of basic problems in the theory of vector fields and in elasticity.]** Izdat. Akad. Nauk. Uzbek. SSR, Tashkent, 1954. 107 pp. 7.90 rubles.

This work discusses a large number of techniques of transforming certain boundary value problems for partial differential equations into integral or integro-differential equations. Some of these are classical, but a good many have recently been devised by the author.

The first chapter is devoted to the determination of a vector field **v** in a domain Q of three-dimensional Euclidean space when its divergence and curl are given in Q and either its tangential components (first problem) or its normal component (second problem) are given on the boundary S of Q; both internal and external problems are discussed. Appropriate regularity conditions on S and on the boundary values are imposed; we omit the details of these.

Space does not permit the detailed discussion of the methods used; we can only describe them in general terms. By using Green's functions or other resolvents of the Dirichlet or Neumann problem, the determination of **v** is reduced to the solution of an integral equation on S; from this is found an integral equation for **v** itself. Further use of Green's functions leads to an explicit solution for **v** in the first problem; in the second problem, one is obtained under more severe restrictions on S. Applications to problems of fluid dynamics, electrodynamics and elasticity theory are indicated.

The second chapter attacks the Lamé equation of static elasticity theory

$$\mu\nabla^2\mathbf{u}+(\lambda+\mu)\text{grad div }\mathbf{u}=\mathbf{0},$$

either the displacement or the surface forces being given on the boundary. An analogue to classical potential theory is built up, the potential function being now a vector field; vector potentials of body deformations, surface deformations, body forces and surface forces are defined. Analogues of the Gauss-Green-Ostrogradsky group of theorems are proved. Various methods are devised for expressing these vector potentials in terms of harmonic functions.

The vector potentials are used to establish numerous different ways of transforming the boundary value problems for the Lamé equation into singular integral equations on the boundary surface; to begin with, the unknown functions are vector fields on the surface.

Further transformations with the aid of Green's functions lead to integral equations for div **u** and curl **u** on the surface, then to non-singular equations for div **u** and curl **u** throughout the body, and finally to integral equations for **u** itself. Heuristic arguments are given to suggest that these last equations are likely to be soluble by successive approximation in a large range of cases.

The third and final chapter constitutes a first attack along similar lines on the dynamical problems of elasticity theory. The partial differential equation considered is now

$$\alpha \text{ grad div } \mathbf{v} - \beta \text{ curl curl } \mathbf{v} - \partial^2\mathbf{v}/\partial t^2 = \mathbf{f}.$$

Laplace transformation with respect to time reduces it to the form

$$\alpha \text{ grad div } \mathbf{u} - \beta \text{ curl curl } \mathbf{u} - \eta^2\mathbf{u} = \mathbf{F}.$$

By arguments similar to those used in the second chapter the problem is converted into the solution of an integro-differential equation, from which systems of integral equations for div **u** and curl **u** are derived. In a final section the author introduces retarded generalised vector potentials, and uses them to reduce the original problem to an integro-differential-difference equation; this is stated to reduce to more familiar equations in certain special cases. *F. Smithies* (Cambridge, England)

4945:

Lippmann, H. Begründung einer auf Kristallplastizität beruhenden mathematischen Plastizitätstheorie. Ing.-Arch. **26** (1958), 187–197.

A stress-strain relation is derived based on the concept of slip, in a manner similar to but not as comprehensive as the treatment by Batdorf and Budiansky [e.g., J. Appl. Mech. **21** (1954), 323–326]. Some additional physical assumptions are stated which, however, are somewhat crude and do not enter into the final equations in any crucial or testable way. The application to large-strain processes does not seem to be justified in view of the implicit assumption of single slip and the neglect of compatibility conditions.

U. F. Kocks (Cambridge, Mass.)

4946:

Yoshimura, Yoshimaru. Comment on the slip theory of Batdorf and Budiansky. Bull. JSME **1** (1958), 109–113.

The author has determined the characteristic shear function of slip theory by tension tests and also by shear tests on the same material and found them to be different. From this he concludes that slip theory is invalid. The accuracy of the experiments is somewhat dubious. However, at this late date such criticism of slip theory is largely irrelevant and does not detract from its value as a contribution to the field of plasticity.

J. L. Sanders (Cambridge, Mass.)

4947:

Fedorov, F. I. The relation between the dielectric permittivity tensor and the stress tensor in an elastically deformed isotropic medium. Akad. Nauk BSSR Trudy Inst. Fiz Mat. **1956**, no. 1, 208–212. (Russian)

Starting with very general considerations of invariance the author obtains the basic equations of photoelasticity (which he calls Maxwell's equations) in an invariant tensor form. This derivation is considered to be of particular interest for the following reasons: (1) The invariant tensor formulation may simplify an optical analysis of the stresses in an arbitrarily oriented sample; (2) a means is provided for establishing a relation between the optical properties and the stress in the case when deformations are not small.

J. E. Rosenthal (Passaic, N.J.)

4948:

Hristoforov, V. V. Construction of the integral equations of the plane theory of elasticity by a method from the theory of vector potentials. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. **10** (1953), no. 2, 134–159. (Russian)

The theory of three-dimensional vector potentials of surface tractions, volume forces, and surface displacements has been used by I. S. Aržanyh [Byull. Sredneaziat. Gos. Univ. **30** (1949)] to reduce to integral equations (for the unknown "densities" of the vector potentials) the solution of the first and second boundary value problems of the static three-dimensional theory of elasticity. Following Aržanyh's general scheme, the author had previously [Trudy Sredneaziat. Gos. Univ., 1952] sketched a theory of two-dimensional vector potentials. In the present paper the details of such a theory are filled in. In particular, the relations between contour vector potentials of traction and deformation on the one hand, and logarithmic contour potentials of simple and double layers on the other, are worked out. A fundamental solution for the plane equations of elasticity employed is derived as a specialization of a general formula of Aržanyh [Dokl. Akad. Nauk UzSSR **1950**, no. 3], which furnishes solutions of the Lamé equations for the displacement vector in three-dimensional elasticity, in terms of harmonic functions. *J. B. Diaz* (College Park, Md.)

4949:

Hristoforov, V. V. Integral equations of plane dynamical problems of the theory of elasticity. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. **13** (1954), 93–102. (Russian)

Employing the properties of plane vector potentials developed in the paper reviewed above, the author derives integral equations satisfied by the displacement vector in the two fundamental boundary value problems of the theory of plane dynamical elasticity. The general scheme is that used by I. S. Aržanyh [Dokl. Akad. Nauk SSSR (N.S.) **76** (1951), 501–503; MR **13**, 88] in the three-dimensional problem. *J. B. Diaz* (College Park, Md.)

4950:

Caĭ, I. P. On variational methods of Leĭbenzon and Ritz for which the coordinate functions may be taken in the form of particular solutions of Lamé's equations. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. **15** (1955), 143–156. (Russian)

The first and second boundary value problems of the theory of elasticity are treated in a general approximate way, in accordance with the variational methods of Trefftz-Leĭbenzon and Ritz [see L. S. Leĭbenzon, Variational methods for the solution of the problems of the theory of elasticity, Moscow-Leningrad, 1943; MR **6**, 79]. The author constructs explicit particular solutions of the Lamé equation $(\lambda+\mu)\text{grad div } u+\mu\nabla^2 u=0$, in spherical coordinates, in the form $u=G+B\text{grad}(r, G)$, where G is a harmonic vector and r is the position vector [P. F. Papkovich; see Leĭbenzon's book], and also in the form $u=H+A\text{rot}[r, H]$, where H is a harmonic vector [I. S. Aržanyh, Dokl. Akad. Nauk UzSSR **1950**, no. 3]. These particular solutions are then employed in outlining the general scheme of the variational methods mentioned in the title. *J. B. Diaz* (College Park, Md.)

4951:

Aržanyh, I. S. Retarded potentials of the dynamic of an elastic body. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. 16 (1955), 5–22. (Russian)

The present paper completes the cycle of considerations initiated by two earlier papers of the author's [Byull. Sredneaziat. Gos. Univ. **30** (1949); same Trudy **8** (1951)] which were concerned with the static problems of elasticity theory, and continued in several papers on the steady vibration problem of elasticity theory [this last was treated in a different manner by V. D. Kupradze, Boundary problems in the theory of vibrations and integral equations, Gostehizdat, Moscow-Leningrad, 1950; Uspehi Mat. Nauk (N.S.) **8** (1953), no. 3(55), 21–74; MR **15**, 318; **18**, 135; **15**, 431]. The central point of the present attack on the dynamic problems of elasticity theory is the derivation of formulas of Kirchhoff's type for the operator

$$\alpha \operatorname{grad} \operatorname{div} - \beta \operatorname{rot} \operatorname{rot} - \frac{\partial^2}{\partial t^2}, \quad \alpha = \frac{\lambda+2\mu}{\rho}, \quad \beta = \frac{\mu}{\rho}.$$

There is a detailed study of the properties of the "body force", "surface displacement", and "surface traction" retarded potentials which occur in the formula of Kirchhoff's type just mentioned. With the aid of these retarded potentials, integro-differential equations satisfied by the solutions of the first and second boundary value problems of dynamic elasticity are then constructed.

J. B. Diaz (College Park, Md.)

4952:

Oniščenko, V. I. The mixed axisymmetrical problem of the theory of potential in a space with a flat circular slit. Dopovidi Akad. Nauk Ukraïn. RSR **1958**, 21–28. (Ukrainian. Russian and English summaries)

The problem is one of finding a pair of harmonic functions $F_1^-(x, y, z)$ and $F_2^-(x, y, z)$ on the lower half-space $z \leqq 0$ and a corresponding pair $F_1^+(x, y, z)$ and $F_2^+(x, y, z)$ on the upper half-space $z \geqq 0$ such that the following boundary conditions are satisfied: $F_1^-(x, y, 0) = G_1(\rho)$, $(\partial F_2^-/\partial z)_{z=0} = G_2(\rho)$, $[\partial F_1^+/\partial z - (1/A)\partial F_2^+/\partial z]_{z=0} = G_3(\rho)$, $F_1^+(x, y, 0) - AF_2^+(x, y, 0) = G_4(\rho)$ for $\rho \leqq a$; and $F_k^+(x, y, 0) = F_k^-(x, y, 0)$, $(\partial F_k^+/\partial z)_{z=0} = (\partial F_k^-/\partial z)_{z=0}$ for $k=1, 2$ and $\rho \geqq a$. A technique of solution is indicated, based on the use of integral transforms.

M. G. Arsove (Seattle, Wash.)

4953:

Duffin, R. J.; and Noll, Walter. On exterior boundary value problems in linear elasticity. Arch. Rational Mech. Anal. 2 (1958), 191–196.

The authors prove uniqueness theorems for three-dimensional equations of classical elasticity for: (a) A region external to closed surfaces when the displacement is prescribed on the surfaces and tends uniformly to zero at infinity; (b) a region external to an infinite cylinder where the (plane) displacement is prescribed on the cylinder and is bounded at infinity. The difference between the results obtained and uniqueness theorems previously known lies in the conditions at infinity.

Uniqueness theorems are also proved for infinite elastic plates bounded internally by a number of smooth curves.

A. E. Green (Newcastle-upon-Tyne)

4954:

Deev, V. M. On the solution of the space problem of the theory of elasticity for anisotropic bodies. Dopovidi Akad. Nauk Ukraïn. RSR **1958**, 707–711. (Ukrainian. Russian and English summaries)

This is a generalization of the Boussinesq-Galerkin solution for elastic isotropic bodies (and Mrs. Mossakowski's solution for rectilinearly orthotropic bodies) to bodies with any rectilinear anisotropy. The three displacement functions Φ_i ($i=1, 2, 3$) satisfy partial differential equations of the sixth order, and the displacements are represented by differential operators of the fourth order over Φ_i.

J. Nowinski (Madison, Wis.)

4955:

Chattarji, P. P. Torsion of curved beams of rectangular cross-section having transverse isotropy. Z. Angew. Math. Mech. **38** (1958), 157–159.

An exact solution is obtained in terms of infinite series involving modified Bessel functions.

W. Freiberger (Providence, R.I.)

4956:

Novinsky, Y. A fundamental assumption of the theory of thin-walled bars with open cross-section. Akad. Nauk Ukraïn. RSR. Prikl. Meh. 3 (1957), 289–294. (Ukrainian. Russian and English summaries)

It is shown that in Vlasov's theory of thin-walled bars [V. Z. Vlasov, "Thin walled elastic bars," Moscow, 1940], it is not necessary to assume the vanishing of the shear strain of the middle surface, and that it is sufficient to suppose the cross-sections of the bar to be rigid in their own planes. In this case, if the ends of the bar are free and if a terminal force is acting, the author shows that Vlasov's formulas give the rigorous solution.

T. P. Anđelić (Belgrade)

4957:

Malyutin, I. S. Longitudinal bending of a rod at the elastic limit. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1958**, no. 8, 112–116. (Russian)

The equation $\partial(pv)/\partial p = F(\partial^3 v/\partial\xi^2\partial p)$ was derived by the author [Izv. Akad. Nauk SSSR Otd. Tehn. Nauk **1957**, no. 12, 43–46] to describe the behavior of a bar under longitudinal compression. This paper provides a solution of this equation through the method of Bubnova-Galerkin. Two special cases are carried through to completion and the results given in graphical form.

R. E. Gaskell (Seattle, Wash.)

4958:

Sumcov, V. S. On the boundary conditions on the end faces of an elastic cylinder. Dopovidi Akad. Nauk Ukraïn. RSR **1957**, 494–498. (Ukrainian. Russian and English summaries)

This paper considers the question of satisfying the boundary conditions on the end faces of a short thick-walled cylinder, when the load is applied to the lateral surface (non-axial-symmetric deformation).

From the author's summary

4959:

Naghdi, P. M. The effect of transverse shear deformation on the bending of elastic shells of revolution. Quart. Appl. Math. 15 (1957), 41–52.

The paper is concerned with axi-symmetrical deformations of shells of revolution, including the effect of transverse shear stress deformations. As in the theory without shear deformation, the basic equations are reduced to two simultaneous second-order differential equations for two dependent variables, except that the present reduction is somewhat less simple. The general solution of this system of equations is discussed by an extension of the method of asymptotic integration of Langer.

E. Reissner (Cambridge, Mass.)

4960:

Arkilic, G. M. Stresses in rotating thin plates with curvilinear boundaries. J. Franklin Inst. **266** (1958), 279–292.

The problem of curvilinear plates rotating at constant speed about an axis in the plane of the plate is considered in this paper. Stress components are expressed in terms of coordinates and in terms of certain potential functions. It is shown that by suitable construction of these functions the stress components can be easily determined.

Author's summary

4961:

Keller, Herbert B.; and Reiss, Edward L. Iterative solutions for the non-linear bending of circular plates. Comm. Pure Appl. Math. **11** (1958), 273–292.

The authors study non-linear von Kármán equations for bending of a thin circular plate under uniform normal pressure. Discussion is mainly concerned with plates clamped at the edges and with zero radial displacement, but analysis is valid for other edge conditions. Solution is by an iterative procedure whose convergence properties are studied by means of integral equations. Method is then applied to finite difference formulation of the differential equations in order to obtain numerical solutions. Numerical results are compared with previous work by other authors and the advantages of the present method are indicated.

A. E. Green (Newcastle-upon-Tyne)

4962:

Weinitschke, Hubertus J. On the nonlinear theory of shallow spherical shells. J. Soc. Indust. Appl. Math. **6** (1958), 209–232.

Der Verfasser geht in der vorliegenden Arbeit von einer nichtlinearen Theorie der Rotationsschalen aus, die von Reissner entwickelt wurde, und wendet sie auf den Fall der flachen Kugelkalotte mit gleichmäßiger Belastung in radialer Richtung an. Am Rande setzt er freie Auflagerung oder starre Einspannung voraus.

In der Theorie von Reissner werden alle Schnittkräfte und Verschiebungen durch zwei unbekannte Funktionen ausgedrückt, von denen die erste eine Spannungsfunktion darstellt, während die zweite die Winkeldrehung der Meridiantangente bedeutet. Sie sind durch ein System von zwei nichtlinearen Differentialgleichungen miteinander verknüpft. Für hinreichend kleine Belastungen kann man die nichtlinearen Glieder streichen und erhält die übliche Theorie als erste Annäherung.

Die Integration dieser Differentialgleichungen erfolgt nach einem von Simons angegebenen Verfahren, indem die beiden unbekannten Funktionen in Potenzreihen nach der Meridiankoordinate entwickelt werden. Man gelangt so zu Rekursionsformeln, die zusammen mit den Randbedingungen ein System von unendlich vielen nichtlinearen Gleichungen für die Koeffizienten ergeben. Nach einem besonderen Iterationsverfahren kann der Verfasser unter Verwendung elektronischer Rechenautomaten dieses System mit jeder gewünschten Genauigkeit lösen.

Er berechnet sodann Last-Verformungskurven für Schalen verschiedener Stichhöhen. Für sehr flache Schalen ergibt sich ein überall eindeutiger Zusammenhang, während für mittlere Stichhöhen innerhalb eines gewissen Bereichs zu ein und derselben Last mehrere Verformungszustände gehören, da die Kurven hier ein Maximum und Minimum aufweisen. Für relativ große Stichhöhen und hohe Belastungen versagt das Rechenverfahren, so daß nur Aussagen im unterkritischen Bereich möglich sind.

Abschließend werden die Ergebnisse mit den Arbeiten von Kaplan, Fung und Archer verglichen, welche dasselbe Problem auf andere Weise behandelt haben.

W. Zerna (Hannover)

4963:

Mittelmann, G. Beitrag zur Berechnung von Translationsschalen. Ing.-Arch. **26** (1958), 288–301.

Der Spannungs- und Verformungszustand von Translationsschalen erscheint theoretisch weitgehend geklärt zu sein. Es fehlten jedoch bisher Untersuchungen über die Lösungen der Gleichungen und damit Verfahren zur Gewinnung numerischer Ergebnisse. Mit dieser Frage befaßt sich die vorliegende Arbeit. In ihrem ersten Teil sind bisher bekannt gewordene Ergebnisse zusammengestellt. Die Differentialgleichungen der Membrantheorie werden nochmals neu abgeleitet, wobei durch geschickte Einführung der Koordinaten und der Belastungsglieder einige Vereinfachungen im Vergleich zu den bekannten Formen erzielt werden konnten. Für den speziellen Fall, daß die Querschnittskurven Parabeln zweiter Ordnung sind, wird die gefundene Differentialgleichung der Membrantheorie für den Lastfall Eigengewicht bei konstanter Schalenstärke gelöst. Der Verfasser setzt sich dann mit der bei Translationsschalen bekannten Schwierigkeit auseinander, daß die bei der Membrantheorie geforderten Randbedingungen in den Ecken nicht ohne weiteres erfüllt werden können. Unter Heranziehung des Formänderungszustandes wird dann eine Näherungslösung vorgeschlagen. Für die maßgebenden Schnittkräfte entlang der Ränder werden Formeln zur überschläglichen Berechnung abgeleitet. Im weiteren Verlauf der Arbeit wird unter Verwendung bekannter Beziehungen das Randstörungsproblem untersucht. Unter Einführung einiger nicht sehr einschränkender Näherungen wird die Berechnung der von den Rändern ausgehenden Störungen im einzelnen verfolgt. Für die Sonderfälle, daß die Schalen in den Randträgern eingespannt oder mit ihnen gelenkig verbunden sind, werden einfache Formeln für die zur Bemessung erforderlichen Biegemomente angegeben. Am Schluß der Arbeit wird ein Zahlenbeispiel vorgeführt und die eingeführten Voraussetzungen und Näherungen dabei zahlenmäßig diskutiert sowie Vergleiche mit anderen Berechnungsverfahren angestellt.

W. Zerna (Hannover)

4964:

Mikeladze, M. Š. Elasto-plastic equilibrium of anisotropic shells. Soobšč. Akad. Nauk Gruzin. SSR **20** (1958), 13–20. (Russian)

Sandwich shells with stressed orthotropic faces and transverse force resisting core are investigated. Elastic and plastic coefficients of the material of the faces are supposed to satisfy a condition which, for isotropic shells, corresponds to the incompressibility of the material. Particular types of states of stress, called "simplest complex states" and characterized by one of four groups of three conditions, such as $\sigma_{xe}=-\sigma_{xi}$, $\tau_{xye}=-\tau_{xyi}$, $\kappa_y=0$, are considered (e and i denoting the outer and the inner faces, respectively, and κ_y the curvature provoked by the load).

A theorem is proved for simplest complex states stating that either elastic or plastic separate regions in the shell, but no mixed elasto-plastic regions, can exist.

An example concerning half-infinite circular tube acted on by terminal shear tractions is worked out in detail, the deflection of the wall in the elastic region being determined in the customary way. In the plastic region, Huber-Mises-Hencky criterion of yielding is applied and,

following Ilyushin, the length of the plastic segment of the tube as well as the deflection of the wall is found. The last problem is equivalent to the solution of a Volterra integral equation. *J. Nowinski* (Madison, Wis.)

4965:

Maunder, L.; and Reissner, E. Pure bending of pretwisted rectangular plates. J. Mech. Phys. Solids **5** (1957), 261–266.

An explicit solution is obtained for the elastic bending of pretwisted rectangular plates subjected to bending moments along two opposite edges. The basic equations are two fourth order linear differential equations in the stress function and the normal displacement. These equations are a special case of shallow shell equations and reduce to the flat plate equations if the pretwist is zero. The solution shows that the stresses and deflections due to the applied bending moment whose axis is normal to the plane of the undistorted plate are appreciably influenced by the pretwist. This is caused by an interaction between the pretwist and the lateral contractions due to bending which is not predicted in classical plate theory. *S. Bodner* (Providence, R.I.)

4966:

Korolevič, Yu. S.; and Grigorenko, Ya. M. On the asymptotic solution of the problem of axisymmetrical deformation of a conical shell with linearly varying thickness. Dopovidi Akad. Nauk Ukraïn. RSR **1958**, 821–825. (Ukrainian. Russian and English summaries)

The axisymmetrical deformation of a conical shell with linearly varying thickness is considered in this paper. By applying asymptotic methods, a solution of the problem is obtained. The accuracy of the method as applied to the problem under consideration is also estimated. The theory is illustrated by a numerical example.

Authors' summary

4967:

Miles, John W. Supersonic flutter of a cylindrical shell. II. J. Aero. Sci. **25** (1958), 312–316.

In part I [same J. **24** (1957), 107–118; MR **18**, 839] the author developed an elegant theoretical approach to the panel flutter problem for a cylindrical shell which may be prestressed in both the axial and circumferential directions, and which may be filled with liquid or gas. In the present, concluding installment, the analysis of part I is completed. Many of the results are presented in graphical or in convenient analytical form.

M. Goland (San Antonio, Tex.)

4968:

Yu, Yi-Yuan. Vibrations of thin cylindrical shells analyzed by means of Donnell-type equations. J. Aero. Sci. **25** (1958), 699–715.

The author derives in an elegant manner the equations for dynamically loaded thin cylindrical shells, with and without consideration of *SR* (transverse shear strain and rotatory inertia). He applies them to non-symmetric vibration and wave propagation in cylinders with several end conditions. Numerical applications to infinite cylinders are included. Much of this work has been presented previously in report form.

The equations are reduced to one equation in the radial displacement w alone and equations relating each of the other displacements to w, a form called "uncoupled" by some, and "Donnell-type" by the author since it was first used by the reviewer in 1933 for developing simplified equations of somewhat limited applicability. As the author stresses, equations in this form are more convenient to apply. {However, in another paper [Proc. Fifth Internat. Congr. Appl. Mech., Cambridge, Mass., 1938, pp. 66–70; Wiley, New York, 1939] the reviewer showed that an even greater advantage of this form is that it permits the importance of the various effects of curvature to be compared on a common basis, in the equation in w. This seems to be the only rational way to make the choice which all authors of practical theories have to make between the multitudinous terms which curvature introduces. In the 1939 paper it is shown on this basis that many of the terms which the author includes (for the case without *SR*, and, hence, probably also for the case with it) are superfluous.} *L. H. Donnell* (Chicago, Ill.)

4969:

Piszczek, Kazimierz. Dynamical stability of plane form of bending with various boundary conditions. Rozprawy Inż. **4** (1956), 175–225. (Polish. Russian and English summaries)

The paper deals with the boundary problem for the equation $(d^2y/dt^2)+(\lambda+\gamma_1\cos t+\gamma_2\cos 2t)y=0$. Using a method analogous to that used for Mathieu's equation, the eigenvalues $\lambda=\lambda(\gamma_1,\gamma_2)$ and some first coefficients of the corresponding eigenfunctions are found. Using Haupt's theorem the "body of dynamical stability" obtained is divided into stable and unstable regions. It is shown that with a suitable choice of the parameters γ_i, the second resonance region does not appear. The tables of Klotter and Kotowski are completed.

The second part of the paper treats the problem of the stability of a bar of narrow rectangular cross-section loaded at the extremities with a moment $M(t)$ and an axial force $P(t)$, of the same angular frequency. Boundary conditions of simple support and other types are considered. K. Klotter's method is used in a modified form.

D. P. Rašković (Belgrade)

4970:

Ivahnin, I. Ĭ. Stability of a conic shell of circular section under uniform compression along the generators. Dopovidi Akad. Nauk Ukraïn. RSR **1958**, 267–271. (Ukrainian. Russian and English summaries)

The problem is solved by Ritz's method and brought to a computational stage.

4971:

Ivahnin, I. Ĭ. Stability of conic shell of circular cross section under the simultaneous action of axial compression and normal external pressure. Dopovidi Akad. Nauk Ukraïn. RSR **1958**, 376–380. (Ukrainian. Russian and English summaries)

4972:

Archer, Robert R. Stability limits for a clamped spherical shell segment under uniform pressure. Quart. Appl. Math. **15** (1958), 355–366.

The boundary value problem

$$\lambda^{-2}L\beta+\psi=-2px+x^{-1}\psi\beta,\quad \lambda^{-2}L\psi-\beta=-\tfrac{1}{2}x^{-1}\beta^2,$$

where $L=d^2/dx^2+x^{-1}d/dx-x^{-2}$ and $\beta(1)=0$, $\psi'(1)-\nu\psi(1)=0$, $\beta(0)$ and $\psi(0)$ regular, is solved by expansions of the form $(\beta,\psi,p)=\sum(\beta_n,\psi_n,p_n)W^n$, where W is a perturbation parameter. Starting with $\beta_1=\sum a_n^{(1)}J_1(\lambda_m x)$, $\psi_1=c_1x+\sum b_m^{(1)}J_1(\lambda_m x)$, where $J_1(\lambda_m)=0$, and imposing an additional physical requirement on certain integrals of the functions β_n, the author is led to analogous expansions for β_n and ψ_n when $n\geqq 2$, and to appropriate

expressions for the quantities p_n. The stability limit is defined as the value of p for which $dp/dW=0$. Numerical calculations are carried out retaining up to five terms in the perturbation series and the results, which go further than results of previous authors obtained by different method, are shown graphically. The paper includes a set of tables of the functions $[\frac{1}{2}J_0^{(2)}(\lambda_n)]^{-1}\int_0^1 J_1(\lambda_i x)J_1(\lambda_j x)\times J_1(\lambda_n x)dx$ for various values of i, j, n.

E. Reissner (Cambridge, Mass.)

4973:

Symonds, P. S.; and Mentel, T. J. Impulsive loading of plastic beams with axial constraints. J. Mech. Phys. Solids **6** (1958), 186–202.

The paper is concerned with the plastic deformation of simply supported and clamped beams whose ends are prevented from displacing axially, and which are subjected to a transverse uniform impulse. During the ensuing motion, there is an axial force N and a bending moment M at any section of the beam. In order for plastic deformation to occur, M and N must satisfy a plasticity condition, and the associated plastic deformation must satisfy the flow rule.

Furthermore, accelerations in the direction of the beam are neglected, so that N is constant along the beam. Under these assumptions, the equations of motion are set up and solved for a plausible mode of motion. The solutions indicate that the deflections are greatly reduced by the action of the axial force.

It is further shown that for large initial velocities, the beam behaves as a plastic string sometime after the impact.

{The reviewer notes that the problem as formulated by the authors has non-unique solutions. It would, therefore, be desirable to learn that the particular mode of motion adopted by the authors has a physical significance, and is the one which will occur.}

E. T. Onat (Providence, R.I.)

4974:

Bishop, J. F. W. On the effect of friction on compression and indentation between flat dies. J. Mech. Phys. Solids **6** (1958), 132–144.

The paper is concerned with the effect of Coulomb friction on compression and identation between flat dies. Mean die pressures are given for different geometrical shapes and for varying constraint, friction and geometric parameters. Whenever possible, the results are obtained from the known exact solutions.

E. T. Onat (Providence, R.I.)

4975:

Kadashevich, Iu. I.; and Novozhilov, V. V. The theory of plasticity which takes into account residual microstresses. J. Appl. Math. Mech. **22** (1958), 104–118 (78–89 Prikl. Mat. Meh.).

In spite of its title, this interesting paper is concerned with a phenomenological description of the strain-hardening behavior of plastic materials. Particular emphasis is placed upon the Bauschinger effect, and the changes in shape and position of the yield surface are related to the strains suffered by the material element. The approach is very similar in spirit to Prager's novel and interesting treatment of the same problem [W. Prager, Proc. Inst. Mech. Engrs. **169** (1955), 41–57; MR **17**, 558]: The mechanical models used in deriving the rules of hardening are almost the same. The present treatment is more general, and the results obtained partly coincide with the recent results of Shield and Ziegler [Z. Angew. Math. Phys. **9a** (1958), 260–276; MR **20** #6859].

E. T. Onat (Providence, R.I.)

4976:

Onat, E. T. Analysis of shells of revolution composed of workhardening material. J. Mech. Phys. Solids **7** (1958), 45–59.

The behavior of a rigid-workhardening material is characterized by the current yield condition and the flow law. In the present paper it is assumed that the initial yield condition is that of maximum shear stress and that the plastic potential flow law is valid. Two possible theories for determining the current yield condition are discussed: isotropic hardening in which no anisotropy is introduced by plastic flow, and a rule elsewhere referred to as kinematic hardening which was originally suggested by Prager. In each case, the current yield condition is fully defined in terms of the strain history of the element and suitable material constants. Specific equations are given for the case of plane stress.

According to the usual assumptions of thin shell theory, strain rates are fully defined by the elongations and curvature rates of the middle surface; the state of stress is defined by force and moment resultants. For the particular case of a rotationally symmetric shell there are four such generalized strain rates, e_θ, e_ϕ, κ_θ, κ_ϕ, and four corresponding generalized stresses N_θ, N_ϕ, M_θ, M_ϕ. The plastic plane stress theories are translated into terms of generalized strain rates and stresses.

The resulting equations are quite complicated. Two applications are made to a circular cylindrical shell without end load. First, a finite, homogeneous, constant strain-rate loading is applied and explicit formulas obtained for the current yield condition for each theory. Then, the incipient plastic flow problem is solved for a simply supported shell under uniform load. The corresponding yield-point load is the same as that for a perfectly plastic material. However, whereas the velocity distribution for the ideal material is not unique, that for the workhardening material is unique.

P. G. Hodge, Jr. (Chicago, Ill.)

4977:

Thomas, T. Y. Plastic disturbances whose speed of propagation is less than the velocity of a shear wave. J. Math. Mech. **7** (1958), 893–900.

The paper considers a first order surface of singularity in a perfectly plastic stress field governed by the Mises yield condition and associated Prandtl-Reuss flow law, a first order singular surface being defined as one across which stress and velocity are continuous but at least one of their first derivatives is discontinuous. Letting $|G-v_n|$ denote the normal velocity of propagation of the surface, the author shows that different results hold for $|G-v_n|=0$, $0<|G-v_n|<(\mu/\rho)^{\frac{1}{2}}$, and $|G-v_n|=(\mu/\rho)^{\frac{1}{2}}$ and considers the middle case in detail {it follows from eq. (2.7) that $|G-v_n|>(\mu/\rho)^{\frac{1}{2}}$ is impossible}. Here, μ is the elastic shear modulus and ρ the density, so that $(\mu/\rho)^{\frac{1}{2}}$ represents the velocity of a shear wave. Several general results are proved, among them the fact that the crack velocity is less than the shear wave velocity if and only if the rate of work of change of shape is different on the two sides of the surface.

Specific results are obtained for the case of plane stress. It is shown that the normal stress across the surface must be zero whereas the corresponding shear stress cannot be zero. Applied to the propagation of a

symmetric crack, this result leads to the conclusion that the shear stress must be singular at the vertex of the crack.

P. G. Hodge, Jr. (Chicago, Ill.)

4978:

Kuznecov, A. I. Plane deformation of non-homogeneous plastic bodies. Vestnik Leningrad. Univ. **13** (1958), no. 13, 112–131. (Russian. English summary)

Plane state of strain in a rigid-plastic body with variable yield point (plastically heterogeneous body) is investigated using Huber-Mises-Hencky yield condition. It is shown that the nonlinear system of basic equations, in contrast to the constant yield point case, cannot be reduced to a linear system by any transformation of variables. Solutions of particular problems are given such as (a) Cauchy problem for a rectilinear boundary, (b) axially symmetrical problem involving a hole in an infinite plane, and (c) generalized Galin's problem (i.e. problem (b) with no axial symmetry). For a slight non-homogeneity of a material the stress is split into two components, the first corresponding to a constant yield point and the second involving a perturbed state. For the latter stress a linear yield condition is obtained, which gives a second order hyperbolic equation for the Airy stress function. An example concerning Prandtl's problem of an indentation of a rigid punch with no friction into a half-plane is solved, assuming the yield stress in the form $k=k_0+cy$, with k_0 constant and c a small factor.

Some of author's results generalize the earlier work by W. Olszak and his collaborators.

J. Nowinski (Madison, Wis.)

4979:

Sharma, Brahmadev. Thermal stresses in transversely isotropic semi-infinite elastic solids. J. Appl. Mech. **25** (1958), 86–88.

The steady-state thermoelastic problem for a half-infinite transversely isotropic half-space, with the axis of rotational elastic symmetry perpendicular to the plane boundary, is considered. The solution of the heat equation being represented by a Fourier integral T, thermoelastic displacement potential is introduced composed of two functions, Φ and $\bar{T}$, the latter represented by a Fourier integral similar to T. Satisfying of the Lamé equations yields a degenerated Laplace equation for Φ which is solved using Fourier integral representation. The general axisymmetric case is also solved involving, in particular, a uniformly heated circular region of exposure.

J. Nowinski (Madison, Wis.)

4980:

Naghdi, P. M. On thermoelastic stress-strain relations for thin isotropic shells. J. Aero./Space Sci. **26** (1959), 125.

It is pointed out that a solution of the thermoelastic problem for isotropic shells, in the spirit of Love's theory of shells, does not conform to the condition of a non-existence of stress in a stress-free and unconstrained shell embedded in a uniform temperature field.

Previous results of E. Reissner for shells of revolution, and of the author for general shells, involving a refined theory of shells, are used in order to establish thermoelastic stress-strain relations free from the defect just mentioned.

J. Nowinski (Madison, Wis.)

4981:

Podstrigač, Ya. S. Thermal field in thin shells. Dopovidi Akad. Nauk Ukraïn. RSR **1958**, 505–507. (Ukrainian. Russian and English summaries)

The author introduces two characteristic temperature functions: one represents the mean value of the temperature over the thickness of the shell and another the mean value of the moment of the temperature. For these characteristic functions there are derived two differential equations, with the corresponding boundary conditions, and the author shows that for any shell of revolution it is possible to construct a solution for the corresponding boundary value problem.

M. Z. v. Krzywoblocki (Urbana, Ill.)

STRUCTURE OF MATTER

See also 4945, 5059

4982:

Maškevič, V. S. The normal coordinates of a crystal lattice with allowance for retarded interaction. Dokl. Akad. Nauk SSSR **121** (1958), 247–249. (Russian)

The analysis of oscillations of a crystalline lattice is usually based on the assumption of a potential energy of interaction which is dependent only on the instantaneous positions of the particles in the lattice. In this case the concept of normal modes of vibration is a direct generalization of the corresponding theory of small vibrations of a finite number of particles.

The author extends this theory to include the case in which the motion of the particles gives rise to an electromagnetic interaction which is propagated with the speed of light in the lattice. This induced field is considered in the electric dipole approximation. Coupled oscillations of the lattice and the field are considered and normal coordinates are introduced as the amplitudes of these oscillations. The expression for the total energy is reduced to Hamiltonian form in terms of the normal coordinates. {For an alternative treatment see pages 89–100 of M. Born and K. Huang [Dynamical theory of crystal lattices, Clarendon Press, Oxford, 1954].}

E. L. Hill (Minneapolis, Minn.)

FLUID MECHANICS, ACOUSTICS

See also 4689, 4705, 4785, 4936, 5020, 5079, 5080, 5083.

4983:

✱Temple, G. An introduction to fluid dynamics. Oxford University Press, New York-London, 1958. xi+195 pp. $4.00; 25 s.

The object of this work is to provide an introduction to fluid dynamics, with the emphasis on physical principles and such developments therefrom as are of practical significance in aerodynamics. The author has designed it for 'students reading for honours in mathematics and theoretical physics', and states that there is an undoubted need for such an introductory work; and the reviewer agrees with this opinion: the work will be valuable not merely for students but also for those who, like himself, were nurtured in the tradition of Lamb's classic treatise, but wish their teaching to be more modern and realistic.

The author considers only fluids that are incompressible and non-viscous; the emphasis is on two-dimensional steady irrotational flows, and of the specific flows that are discussed, the most complicated are the 'disturbance

flow' produced by a circular cylinder in a uniform stream and flows produced from this by conformal mapping, and the discontinuous flow around a flat plate. A final chapter on slender-body theory is a welcome and novel feature. The reader is assumed to be familiar with the calculus of functions of several variables and with the elements of complex-variable theory and vector analysis. The modern point of view of the author is illustrated by his terminology: he speaks nearly always of the 'mass equation' rather than 'equation of continuity' and of 'potential flow' as often as 'irrotational flow', and the term 'Laplace's equation' is never used.

{There are a few trivial misprints, and some details on which the reviewer's taste differs from the author's. For example, the reviewer would say either less or more about Green's theorem, he would give the uniqueness theorem for the Neumann problem, and at various places he would emphasize that the motions considered are steady; also he regards as unfortunate the phrase 'plane parallel motion' — which is perhaps a lapse from 'parallel plane motion' (which occurs earlier in the book with equivalent sense). However, these are minor spots on a book which is well planned and engagingly written.}

T. M. Cherry (Melbourne)

4984:

Pyhteev, G. N. Determination of the axially symmetric potential motion of an incompressible fluid from given values of the direction of its velocity. Bul. Inst. Politehn. Iași (N.S.) 2 (1956), no. 3–4, 35–38. (Russian. Romanian and English summaries)

Let z, r be cylindrical coordinates, $V(z, r)$ the speed of flow, and $\delta(z, r)$ its inclination to the axis of symmetry. Then

$$\partial \ln V/\partial r - \partial\delta/\partial z = r^{-1} \sin^2 \delta,$$
$$\partial \ln V/\partial z + \partial\delta/\partial r = -r^{-1} \sin \delta \cos \delta.$$

Let $\zeta = z + ir$, set the complex potential $\phi + i\psi = w(\zeta, \bar\zeta)$, and let $f(\zeta, \bar\zeta) = i\delta + \ln(V/\alpha)$, where α is a real constant. Then $w(\zeta, \bar\zeta)$ can be found in terms of integrals with respect to ζ or $\bar\zeta$ of products of e^f or $e^f\partial f/\partial\zeta$ by linear functions of $\zeta - \bar\zeta$.

J. H. Giese (Aberdeen, Md.)

4985:

Kroškin, M. G. Some questions on the hydromechanics of a ship. Trudy Morsk. Gidrofiz. Inst. 10 (1957), 53–72. (Russian)

4986:

Dressler, R. F. Unsteady non-linear waves in sloping channels. Proc. Roy. Soc. London. Ser. A 247 (1958), 186–198.

It is found that the shallow water equations for flow in an open channel with uniform slope can be transformed into the corresponding equations for a horizontal channel. The equations for the depth y and the velocity u are

$$u_t + uu_x + gy_x = gm, \quad y_t + yu_x + uy_x = 0$$

(where m is the constant slope), and the transformation uses $\zeta = x - \frac{1}{2}mgt^2$, $w = u - mgt$ in place of x and u. This theory is applied to the dam-break problem for water initially held behind a vertical wall on uniformly sloping ground. The solution employs the hodograph transformation to linearize the equations for $w(\zeta, t)$, $y(\zeta, t)$.

The same transformations were used by Carrier and Greenspan [J. Fluid Mech. 4 (1958), 97–109; MR 20 # 2945] in their discussion of waves on sloping beaches.

G. B. Whitham (New York, N.Y.)

4987:

★de Beaumont, Henry du Boscq. Étude critique de certains prolongements de la mécanique rationnelle: thermodynamique et viscosité. Préface de M. Aubert. Publ. Sci. Tech. Ministère de l'Air, Paris, Notes Tech. no. 72 (1957). viii+47 pp. 1000 francs.

The main purpose of this article is to prove two statements in the theory of viscous fluids. (a) Steady motions with rotation, about an axis of symmetry, in general contain discontinuities in the limit as the viscosity tends to infinity. (b) The force on an obstacle in a steady two-dimensional Stokes flow is zero.

These results appear to be in contradiction to the experience of other workers in the theory of viscous flow and, in the opinion of the reviewer, are wrong.

With respect to the first statement the author first examines the motion engendered by two rotating coaxial circular cylinders, and shows by energy considerations that there can be no motion in the direction of the axis and that, if the region occupied by the fluid includes the axis, only a solid body rotation is possible. {This is all very reasonable and not unexpected since there is nothing driving the fluid in a direction parallel to the axis. If there were, of course, it would do work modifying the energy equation.} The author immediately deduces (a), which appears to be a non sequitur.

In the discussion of the second statement the fact that other writers, notably G. B. Jeffrey [Proc. Roy. Soc. London Ser. A 101 (1922), 169–174], have found a non-zero drag is ignored or else their work is labelled as erroneous. The proof depends on showing that if $\nabla^4\psi = 0$, and the Laurent expansion of ψ in an annular region, which does not include the origin $r=0$, contains the term (1) $C_{-1}r \cos(\theta - \theta_{-1})\log r$, C_{-1}, θ_{-1} constants, then ψ is singular at $r=0$.

But one can give a counterexample. For consider

$$\psi = (r \cos \theta - a)\log(r^2 - 2ar \cos \theta + a^2).$$

This function satisfies $\nabla^4\psi = 0$, and its Laurent expansion in $r \geq b > a$ certainly includes a term of the form (1), but it is clear that ψ is well behaved at $r=0$.

K. Stewartson (Durham)

4988:

Tamada, K.; and Fujikawa, H. The steady two-dimensional flow of viscous fluid at low Reynolds numbers passing through an infinite row of equal parallel circular cylinders. Quart. J. Mech. Appl. Math. 10 (1957), 425–432.

A summary of extensive calculations for the plane boundary value problem: $[\nabla^2 - (U/\nu)\partial/\partial x]\zeta = 0$; $w \equiv u - iv = 0$ on the circumferences of a vertical row of circles with radius a and central distance h, $w = U$ for $|x| = \infty$. This formulation is Oseen's approximation; the vorticity $\zeta = 2i\partial w/\partial\bar z$; the method is an extension of Tomotika's and Aoi's work [same Quart. 3 (1950), 140–161; 4 (1951), 401–406; Mem. Coll. Sci. Univ. Kyoto Ser. A 26 (1950), 9–19; MR 12, 59; 13, 699, 397]. The complex velocity w is again sought in the form of a sum of an analytic and a non-analytic part, $\sum_s \sum_m A_m(a/z_s)^m$ and $\sum_s \exp kx_s \sum_m a_m K_m(kr_s) \exp im\vartheta_s$, which takes care of the periodicity of the boundary condition. Here $z_s = z - ish$, $x_s = \mathrm{Re}\,(z_s)$, $r_s = |z_s|$, K_m is the modified Bessel function, and $k = U/2\nu$. The recurrence relations for A_m and a_m are given.

Procedures for small ka are different from those for small ka and kh. The drag per unit length of the row is given for the Reynolds-number range $0 \leqq 2aU/\nu \leqq 8$ for a

set of values of $2a/h$. The drag does not approach zero for vanishing Reynolds number.

G. Kuerti (Cleveland, Ohio)

4989:

Azpeitia, Alfonso Gil; and Newell, Gordon Frank. Theory of oscillation type viscometers. III. A thin disk. Z. Angew. Math. Phys. **9a** (1958), 97–118.

[For earlier parts see J. Kestin and Newell, same Z. **8** (1957), 433–449; MR **19**, 1218; and D. Beckwith and Newell, ibid. **8** (1957), 450–465; MR **19**, 1218.] This paper is concerned with the development of a theory which includes edge effects for a disk of small thickness (measured relative to the boundary layer thickness) and of large radius (measured relative to the boundary layer thickness) performing torsional oscillations in a viscous fluid. First the case of a disk of zero thickness is studied by introducing a coordinate measured from the edge of the disk, and then letting the radius of the disk approach infinity. The resulting semi-infinite geometry problem is solved by the Wiener-Hopf method. A thickness correction is obtained by mapping the finite disk of infinite radius into an infinite disk of zero radius. Thus, the method of approximations is essentially a first correction for a small thickness to a disk of infinite radius and then a first correction to this result for a large but finite radius. The accuracy of the final results is improved by expressing the important quantity (for the viscometry problem) as an integral which has a stationary value.

R. C. DiPrima (Troy, N.Y.)

4990:

Sl'ozkin, M. O. On the theory of the initial space of a plane laminar jet of liquid. Dopovidi Akad. Nauk Ukraïn. RSR **1958**, 702–706. (Ukrainian). Russian and English summaries)

A jet of incompressible viscous liquid flows into a semispace occupied by the same liquid. The motion in the entire domain is assumed to be regular and plane parallel. Linearized equations are employed which take into consideration the partial summands of the acceleration and viscosity. Laplace's equation for the pressure is solved by Fourier's method. The equation for the principal velocity component is solved by the method of operational calculus. The length of the jet core of almost constant velocities is determined in finite form for the case when the pressure in the inflow cross section is equal to the pressure at infinity. *Author's summary*

4991:

Bourne, D. E.; and Davies, D. R. Heat transfer through the laminar boundary layer on a circular cylinder in axial incompressible flow. Quart. J. Mech. Appl. Math. **11** (1958), 52–66.

Les auteurs partent des équations de R. Seban et R. Bond [J. Aero. Sci. **18** (1951), 671–675] avec les corrections introduites par H. Kelly [ibid. **21** (1954), 634] qui sont valables pour des petites valeurs de $\nu x/Ua^2$ où U est la vitesse uniforme du courant moyen, x est la distance du courant descendant, ν la viscosité cinématique et a le rayon du cylindre.

Les auteurs déduisent en utilisant les recherches de Glauert et Lighthill [Proc. Roy. Soc. London Ser. A **230** (1955), 188–203; MR **16**, 1171] une solution asymptotique valable pour des grandes valeurs de x.

Les expressions représentent des séries entières en β^{-1}, $\beta=\mathrm{Ln}(4\nu x/ua^2)$, jusqu'à terme β^{-3}. La fonction du courant est de la forme

$$\psi=f_0+\frac{f_1}{\beta}+\frac{f_2}{\beta^2}.$$

Des expressions analogues sont obtenues pour la vitesse et la température.

En utilisant les résultats obtenus précédemment [Davies and Bourne, Quart. J. Mech. Appl. Math. **9** (1956), 457–467; MR **18**, 777] les auteurs obtiennent une solution approximative intermédiaire entre les deux solutions extrêmes.

M. Kiveliovitch (Paris)

4992:

Tetervin, Neal. A discussion of cone and flat-plate Reynolds numbers for equal ratios of the laminar shear to the shear caused by small velocity fluctuations in a laminar boundary layer. NACA Tech. Rep. no. **4078** (1957), 25 pp.

A comparison is made between the Reynolds number R_c characterizing the flow past a conical surface and the Reynolds number R_p of a flow past a flat plate under conditions of equal closeness to the transition from laminar to turbulent flow. The approximation used is the linear theory of boundary-layer stability augmented by Schlichting's calculated amplification ratios for incompressible flow [Australian Aero. Res. Labs. Transl. no. **6** (1944)]. The chief result is that $R_c-R_p=2R_p^*$, where R_p^* denotes the minimum critical Reynolds number for the plate. Thus the ratio R_c/R_p at transition is equal to three only when transition takes place at or near R_p^*, while this ratio approaches unity when transition occurs at a large multiple of R_p^*.

H. C. Kranzer (New York, N.Y.)

4993:

Craya, Antoine. Contribution à l'analyse de la turbulence associée à des vitesses moyennes. Publ. Sci. Tech. Ministère de l'Air, Paris. **345** (1958), 111 pp.

L'auteur considère un écoulement turbulent homogène non isotrope pourvu d'une vitesse moyenne non nulle $\bar{u}_l$ telle que les quantités $\partial\bar{u}_l/\partial x_m=\lambda_{lm}$ soient constantes. Il se propose d'étudier les divers tenseurs de corrélation de vitesse dans cet écoulement, la réduction du nombre de leurs composantes, leur mesurabilité expérimentale, et leurs ordres de grandeurs numériques.

Conformément aux techniques traditionnelles, il commence par les corrélations doubles $R_{ij}(r)$ en deux points. Il établit l'équation, généralisation de l'équation de Karman-Howarth, qui relie les R_{ij} aux corrélations triples (en deux points), et la traduit en termes spectraux. Il introduit ensuite les corrélations triples en trois points $R_{ijk}(\mathbf{r}, \mathbf{r}')=\overline{u_iu_j'u_k''}$, où u_i, u_j', u_k'' sont les vitesses en trois points M, M', M''.

Des équations tensorielles relient les R_{ijk} au tenseur des corrélations quadruples $Q_{lijk}(\mathbf{r},\mathbf{r}')=\overline{u_lu_iu_j'u_k''}-\overline{u_lu_i}\,\overline{u_j'u_k''}$ et au tenseur de corrélation de la pression et des vitesses en M' et M''. Ces équations sont traduites en termes spectraux, faisant intervenir le tenseur $\phi_{ijk}(\mathbf{k}, \mathbf{k}')$, transformé de Fourier de $R_{ijk}(\mathbf{r}, \mathbf{r}')$. Après quelques remarques sur l'hypothèse de quasinormalité et les tenseurs de déformation, l'auteur étudie (chapitre II) les simplifications que l'incompressibilité et diverses symétries apportent à la structure des ϕ_{ijk}. $\mathbf{k}''$ étant un vecteur spectral défini par $\mathbf{k}+\mathbf{k}'+\mathbf{k}''=0$, il introduit trois trièdres trirectangles de référence $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$, $(\boldsymbol{\alpha}', \boldsymbol{\beta}', \boldsymbol{\gamma}')$, $(\boldsymbol{\alpha}'', \boldsymbol{\beta}'', \boldsymbol{\gamma}'')$, $\boldsymbol{\alpha}'$, $\boldsymbol{\alpha}$, $\boldsymbol{\alpha}''$ étant dirigés respectivement suivant $\mathbf{k}$, $\mathbf{k}'$, $\mathbf{k}''$, et $\boldsymbol{\beta}$, $\boldsymbol{\beta}'$, $\boldsymbol{\beta}''$ étant dans le plan de $\mathbf{k}$, $\mathbf{k}'$, $\mathbf{k}''$. Il montre que les 27 composantes ϕ_{ijk} s'expriment à l'aide de 8 scalaires seulement par les formules

$$\phi_{ijk}=\Gamma\beta_i''\beta_j\beta_k'+\psi\beta_i''\beta_j\gamma_k+\psi'\beta_i''\gamma_j\beta_k'+\Psi''\gamma_i\beta_j\beta_k'+\\ \Theta\gamma_i\gamma_j\beta_k'+\Theta'\gamma_i\beta_j\gamma_k+\Theta''\beta_i''\gamma_j\gamma_k+\Omega\gamma_i\gamma_j\gamma_k.$$

Les symétries générales réduisent ces 8 scalaires à 4 scalaires indépendants. Si la turbulence est isotrope ou si ϕ_{ijk} a un "centre de symétrie", de nouvelles réductions se produisent. Toute cette technique de réduction est enfin appliquée aux corrélations doubles. On trouve dans le cas général que le tenseur spectral correspondant $\phi_{ij}(\mathbf{k})$ s'exprime à l'aide de 3 scalaires suivant la formule

$$\phi_{ij}(\mathbf{k})=N_1\beta_i\beta_j+S\beta_i\beta_j+S^*\gamma_i\beta_j+N_2\gamma_i\gamma_j,$$

où N_1 et N_2 sont des fonctions paires de $\mathbf{k}$, et où $S(-\mathbf{k})=-S^*(\mathbf{k})$.

Le chapitre III débute par une discussion physique du rôle des termes qui, dans l'équation des corrélations doubles, traduisent l'inhomogénéité des écoulements généraux. Ensuite est posé le problème du calcul des ϕ_{ij} en fonction des corrélations triples en deux points, des ϕ_{ijk} en fonction des corrélations quadruples. Des axes locaux sont définis dans l'espace des $\mathbf{k}$, et permettent de représenter commodément les composantes ϕ_{ij}. Trois équations réduites relient, aux composantes réduites des tenseurs de corrélations triples en 2 points supposés connus, les 3 scalaires N_1, S, N_2 qu'il s'agit de calculer. Des indications sur la méthode de résolution sont données dans le cas d'une déformation pure plane, dans le cas d'une turbulence stationnaire (dont l'existence physique n'est d'ailleurs pas certaine), et, enfin, dans le cas d'un écoulement parallèle.

Au chapitre IV, le problème est repris pour les corrélations triples en 3 points. Les équations fondamentales s'écrivent

$$\frac{\partial\phi_{ijk}}{\partial t}+\Psi_{ijk}=\Omega_{ijk}-\nu(k^2+k'^2+k''^2)\phi_{ijk},$$

où

$$\psi_{ijk}=2\lambda_{lm}A_{ijklm}-\lambda_{lm}B_{ijklm},$$
$$A_{ijklm}=\Delta_{il}''\phi_{mjk}+\Delta_{jl}\phi_{imk}+\Delta_{kl}'\phi_{ijm},$$
$$B_{ijklm}=\frac{\partial}{\partial k_m}(k_l\phi_{ijk})+\frac{\partial}{\partial k_m'}(k_l'\phi_{ijk})+\delta_{il}\phi_{mjk}+\partial_{jl}\phi_{imk}+\delta_{kl}\phi_{ijm},$$
$$\Omega_{ijk}=k_l''\Delta_{im}''\Theta_{lmjk}+k_l\Delta_{jm}\Theta_{lmki}'+k_l'\Delta_{km}'\Theta_{lmij}'',$$

et où Θ_{lijk} est le transformé de Fourier du tenseur de corrélations quadruples Q_{lijk} défini ci-dessus. On a posé $\Delta_{ij}=\delta_{ij}-k_ik_j/k^2$. Tous ces tenseurs se représentent comme combinaisons de scalaires de base, entre lesquels on a 4 équations, à la place des 27 équations en ϕ_{ijk}. Le chapitre IV est essentiellement consacré au long calcul des scalaires de base des ψ_{ijk}, A_{ijklm}, B_{ijklm}. En ce qui concerne les Ω_{ijk}, on utilise accessoirement l'hypothèse classique de quasi-normalité, suivant laquelle

$$\Theta_{lmjk}(\mathbf{k},\mathbf{k}')=\phi_{lj}(\mathbf{k})\phi_{mk}(\mathbf{k}')+\phi_{lk}(\mathbf{k}')\phi_{mj}(\mathbf{k}).$$

On commence par le cas isotrope où $\phi_{ij}=\phi\Delta_{ij}$, et on passe ensuite au cas général. On arrive ainsi à une représentation du tenseur $\Omega_{ij}=k_l\Delta_{im}\Theta_{lmj}+k_l\Delta_{jm}\Theta_{lmi}^*$, qui règle l'évolution des corrélations doubles.

Le chapitre V est consacré à l'étude de diverses moyennes en un point, et pour commencer à celle des tensions de Reynolds. On met en évidence, dans les équations de Reynolds, le rôle de l'homogénéité. Les relations entre les corrélations pression-vitesse et les corrélations doubles et triples en 2 points sont examinées. On passe ensuite à la dissipation d'énergie dont l'étude est compliqué par l'anisotropie manifeste de la turbulence. Elle met en jeu les 45 composantes distinctes du tenseur $D_{ijlm}=\overline{\dfrac{\partial u_i}{\partial x_l}\dfrac{\partial u_j}{\partial x_m}}$, qu'il n'est pas possible de mesurer toutes. Lorsqu'il existe un trièdre trirectangle dont les faces sont des plans de symétrie de la turbulence, il reste 9 composantes indépendantes. Les corrélations de tourbillon satisfont à des équations d'évolution analogues à celles des tensions de Reynolds. Les ordres de grandeurs des divers termes, et leur influence sur l'anisotropie de la turbulence sont discutés, ainsi que les moyens directs ou indirects de les mesurer. On passe enfin à l'étude des moyennes $\overline{\dfrac{\partial u_i}{\partial x_l}\dfrac{\partial u_j}{\partial x_m}\dfrac{\partial u_k}{\partial x_n}}$, au nombre de 165. On en fait une classification, puis une réduction tenant compte des symétries, de l'incompressibilité et de l'homogénéité. Dans le cas où la turbulence a trois plans rectangulaires de symétrie, il en subsiste à priori 45 indépendants. L'incompressibilité les réduit à 30 et l'isotropie à 1. Ces corrélations se déduisent des dérivées des corrélations triples de vitesse en trois points, et certaines combinaisons d'entre elles peuvent même se déduire du tenseur des corrélations triples de vitesse en deux points seulement. Le chapitre V se termine par diverses remarques sur le bilan de production et de dissipation du tourbillon ω_i. Si l'on pose

$$q^2=\overline{u_1^2}+\overline{u_2^2}+\overline{u_3^2},\quad \omega^2=\overline{\omega_1^2}+\overline{\omega_2^2}+\overline{\omega_3^2},\quad \frac{q^2}{\omega^2}=\frac{\lambda^2}{5},$$

des considérations de similitude montrent que, en turbulence isotrope et en l'absence de vitesses moyennes, les rapports $q^2(t-t_0)/\nu$ et $q\lambda/\nu$ sont constants. Ces relations sont discutées, puis le problème de l'anisotropie de la turbulence homogène, sans, puis avec des vitesses moyennes, est discuté du point de vue des ordres de grandeurs et de la structure physique de la turbulence.

J. Bass (Paris)

4994:

✱Ferrari, Carlo. Turbolenza di parete. Corso sulla teoria della turbolenza, Vol. 1, pp. 171–286. Centro Internazionale di Matematica Estivo. Libreria Editrice Universitaria Levrotto e Bella, Turin, 1957. viii+339 pp.

This paper provides a comprehensive and up-to-date review of the theory of the turbulent boundary layer, and contains much new analysis of the velocity distributions in inner and outer regions and of the corresponding shear stress and temperature distributions, based on simple models of the eddy structure. Such theories are necessarily tentative and results from them can be accepted only after experimental verification; for this reason a more empirical starting point is usually chosen, and it is still rather early to use the phenomenological theories of turbulence as a basis for analysis such as that in the paper, although the details are interesting in themselves. A somewhat surprising omission is the lack of any reference to the work of Coles [J. Fluid Mech. 1 (1956), 191–226; MR 18, 355; and other papers].

D. A. Spence (Farnborough)

4995:

✱Deissler, Robert G.; and Perlmutter, Morris. An analysis of the energy separation in laminar and turbulent compressible vortex flows. 1958 Heat transfer and fluid mechanics institute, held at University of California, Berkeley, Calif., June, 1958: preprints of papers, pp. 40–53. Stanford University Press, Stanford, Calif., 1958. viii+264 pp. $7.50.

The velocity and temperature distributions in a viscous vortex with radial and axial flow, such as occurs in the Ranque-Hilsch vortex tube, are considered. Analysis is based on the compressible flow equations of motion and

energy and includes the inertia terms, viscous shear stress, energy convertion, viscous dissipation and conduction, and is for both laminar and turbulent flows, including also, in the latter case, an effect due to the expansion and contraction of turbulent eddies as they move radially.

In the laminar case the energy separation is found to be greatest at low Reynolds numbers, cold gas being removable from the centre of the vortex and hot gas from the periphery. In turbulent flow the effect is more marked as the Reynolds number based on the eddy viscosity is always low.

Good agreement with experimental results is reported.
D. A. Spence (Farnborough)

4996:

Birkhoff, Garrett; and Kampé de Fériet, J. Kinematics of homogeneous turbulence. J. Math. Mech. **7** (1958), 663–703.

Ce travail est un exposé mathématique rigoureux des principes de la théorie des champs vectoriels aléatoires qui sont utilisés dans la théorie de la turbulence. La section A contient les définitions et les théorèmes généraux, énoncés dans le langage de la théorie de la mesure: définition d'un champ vectoriel aléatoire par une mesure μ sur l'ensemble F des sous ensembles mesurables d'un ensemble Ω; concept de mesure régulière sur un espace topologique Ω (obtenue par complétion à partir d'une mesure définie sur le corps des ensembles de Borel de Ω) de L; mesure sur un espace de Banach B; construction des mesures régulières dans l'espace $C(D)$ des fonctions continues sur un compact D de R^p, ou dans l'espace $L_2(D)$ des fonctions de carré sommable sur un domaine borné D de R^p. Dans l'espace Λ des fonctions de carré sommable sur des ensembles compacts, on appelle mesure admissible toute mesure régulière ayant une énergie finie sur tout compact D. On étudie enfin les mesures strictement ou métriquement régulières. On définit un champ vectoriel aléatoire mesurable dans $R^p\times\Omega$, et on suppose son énergie moyenne finie sur tout compact D. Toute mesure admissible sur Λ engendre un champ vectoriel aléatoire mesurable. On montre enfin que toute mesure, régulière sur l'espace de Hilbert $H=[L_2(D)]^q$, d'un champ aléatoire ayant une énergie finie sur D est "essentiellement compacte". Enfin ces principes abstraits sont confrontés avec les bases usuelles de la théorie de la turbulence, telles qu'elles sont exposées par G. K. Batchelor [The theory of homogeneous turbulence, Univ. Press, Cambridge, 1953; MR **14**, 597].

La section B est consacrée aux covariances. $u(x, \alpha)$ est un champ à q composantes, défini sur $R^p\times I$, où I est l'intervalle $0\leq\alpha\leq 1$. L'énergie moyenne est supposée finie, et la vitesse moyenne existe, car $\mu(\Lambda)=1$. On peut, en turbulence homogène, la supposer nulle. On introduit la matrice de covariance

$$\Gamma_{jk}(x, y)=\int_\Lambda u_j(x, \alpha)u_k(y, \alpha)d\mu(\alpha)$$

dont les propriétés élémentaires sont classiques. On peut lui associer le système d'équations intégrales

$$\int_D \sum_k \Gamma_{jk}(x, y)\phi_k(y)dm(y)=\lambda\phi_j(y)$$

qui, sur tout compact D, a une base dénombrable de fonctions propres $\phi_{nj}(x)$ et des valeurs propres $\lambda_n\geq 0$. $\Gamma_{jk}(x, y)$ est donc représentable dans $[L_2(D\times D)]^q$ par la série $\sum_n \lambda_n\phi_{nj}(x)\phi_{nk}(y)$. Réciproquement, une série de cette nature représente une matrice de covariances. Le cas des covariances continues est spécialement utile pour les applications, et fournit des champs aléatoires continus en moyenne. On définit ensuite un champ aléatoire normal (ou gaussien) et on montre que les résultats énoncés pour des domaines compacts D s'étendent à des espaces infinis moyennant le théorème d'unicité suivant. Si une matrice de covariance est définie sur tout compact, c'est la matrice de covariance d'un champ aléatoire admissible normal unique. La section B se termine par une discussion des conditions d'homogénéité, isotropie, incompressibilité, irrotationnalité. La matrice de covariance $R_{jk}(h)=\Gamma_{jk}(x, x+h)$, ou matrice de corrélation, d'un champ aléatoire homogène admissible est uniformément continue en moyenne. Elle peut d'ailleurs ne jamais être continue localement. Un example est donné de fonction aléatoire discontinue partout, mais continue en moyenne.

La section C est consacréé à l'étude mathématique du spectre d'énergie de la turbulence homogène: rappel du théorème de Bochner; extension à la représentation spectrale d'une matrice de corrélation. Exemples: étude des champs isotropes et expression de l'incompressibilité en termes spectraux. Dans le cas de l'isotropie, un théorème de Schoenberg permet d'exprimer la fonction $\rho(|h|)=\sum R_{jj}(h)$ comme transformée de Lebesgue-Stieltjes-Hankel d'une mesure spectrale scalaire $\phi(k)$. Enfin quelques propriétés des spectres absolument continus et de la turbulence homogène et isotrope à trois dimensions sont examinées.
J. Bass (Paris)

4997:

Jain, P. C. Density fluctuations in turbulence in an inviscid compressible fluid. Proc. Nat. Inst. Sci. India. Part A. **24** (1958), 40–44.

Equations for the density covariance,

$$\bar{\omega}=\overline{(\rho'-\bar{\rho})(\rho''-\bar{\rho})}$$

(primes refer to values at positions x_i' and x_i'' and times t' and t'') are derived for an inviscid compressible fluid in isotropic turbulent motion. Using the equation expressing conservation of mass, it is shown that

$$\int_0^\infty r^2\bar{\omega}(r)dr=\text{const.}\quad (r=|x_i'-x_i''|)$$

for all t' and t'' if the scalar function defining the first-order tensor, $\overline{\rho''\rho'u_i'}=L(r)r_i$, falls to zero more rapidly than r^{-3}. (Note, however, that if $\bar{\omega}\to 0$ as $t'-t''\to\infty$ for all r, the value of the constant is necessarily zero.)

Using the Navier-Stokes equation for an inviscid fluid and assuming (a) that the turbulence is stationary in time, (b) that the fourth-order correlation $\overline{\rho''\rho'u_i'u_j'}$ is related to the second-order correlations as in a joint Gaussian distribution, and (c) that variations in pressure and density are adiabatic, it is shown that $\bar{\omega}(r, t)$ satisfies the wave equation with wave-velocity $(c^2+\frac{1}{3}\overline{u_i^2})^{\frac{1}{2}}$,

$$\frac{\partial^2\bar{\omega}}{\partial t^2}=(c^2+\tfrac{1}{3}\overline{u_i^2})\left(\frac{\partial^2\bar{\omega}}{\partial r^2}+\frac{2}{r}\frac{\partial\bar{\omega}}{\partial r}\right).$$

The wave-velocity differs from that previously found by Chandrasekhar. *A. A. Townsend* (Cambridge, England)

4998:

Murray, James D. Two dimensional compressible shear flow in a channel. Quart. Appl. Math. **15** (1957), 231–236.

The steady two-dimensional flow of an inviscid compressible fluid under conditions of adiabaticity, rotationality, and homentropy is studied. The particular flow discussed is an initally parallel shear flow of constant

velocity gradient (necessarily then of constant initial vorticity) through a divergent channel with straight parallel sides far upstream and downstream of the expansion of the channel to a channel of double width. Attention is directed to conditions at the entrance and exit sections and, under assumptions of sufficiently parallel flow at these sections, expressions are obtained for the final density, vorticity, and stream function distribution. Quantitative results are obtained and it is shown that for completely subsonic flow the final velocities are less, the density greater, vorticity greater, and the deflection of the initially central stream line is downward. Opposite results are valid for the completely supersonic case. The stream line deflection for the supersonic case is of an order of magnitude less than that for the subsonic case. This result has been observed experimentally.

P. Chiarulli (Chicago, Ill.)

4999:

Gundersen, Roy. The flow of a compressible fluid with weak entropy changes. J. Fluid Mech. 3 (1958), 553–581.

The first order non-homentropic perturbation of the one-dimensional homentropic basic flow (b.f.) of a perfect gas is explicitly determined for the following cases: the b.f. is uniform; a centered simple wave; a general simple wave, but the first order (f.o.) perturbation still homentropic; and the b.f. itself being a first order perturbation of a uniform flow. The general solutions for the f.o. perturbations of u and c (particle and sound velocity) depend on the distribution of the f.o. disturbance-entropy across the particle-lines of the b.f. This method is applied to the following problems: the shock produced by a uniformly moving piston, the uniformity being slightly disturbed for $t>t_1$; the usual shock tube problem for slightly disturbed initial (rest) conditions on both sides of the diaphragm; and a uniform shock moving into a tube-section of slowly-varying cross section.

In the present work the "physical" variables x and t are the independent variables. Problems of the same general nature have been attacked earlier by P. A. Fox [J. Math. Phys. **34** (1955), 133–151; MR **17**, 208] and also by S. C. Himmel [NACA Tech. Note **TN 1439** (1958)], who use perturbation series for u, c, x, t in terms of two characteristic parameters, thus assuming the perturbation still as homentropic.

G. Kuerti (Cleveland, Ohio)

5000:

Belen'kii, I. M. Quasi-stationary flow of a gas from a cylindrical container of variable volume. J. Appl. Math. Mech. 22 (1958), 383–392 (279–285 Prikl. Mat. Meh.).

A piston moves in a cylinder whose end has a small opening. Gas is continually being introduced into the cylinder and escapes through the opening, at which there is no back pressure. It is assumed that quasi-steady conditions have been attained, that is, all the variables of state are functions of time alone, that the gas speed in the cylinder is negligible, and that the mass inflow (when not zero) is proportional to the pressure. The mutual relations between piston speed and the variables of state are investigated for three special cases: (i) piston stationary; (ii) equal mass inflow and outflow; (iii) mass inflow cut off.

H. C. Levey (Nedlands)

5001:

Korobeĭnikov, V. P. The exact solution of a nonlinear problem involving an explosion in a gas of variable initial density. Dokl. Akad. Nauk SSSR (N.S.) **117** (1957), 947–948. (Russian)

Let $\nu=1, 2, 3$ for one-dimensional cylindrically symmetric, or spherically symmetric flow; let $2k=\nu(\gamma-1)+2$; and let r be the space coordinate. The author exhibits for the corresponding one-dimensional unsteady flow equations a solution of the type

$$v=r/kt,\ p=c_1(kt)^{-\gamma\nu/k}F(x),\quad \rho=c_2(kt)^{-(\nu-1)/k,-1}dF/dx,$$

where $x=r(kt)^{-1/k}$, c_1 and c_2 are certain constants, and F is described by the solution of a certain functional equation. It is possible to fit this solution to a shock at $r=r_2(t)$ advancing into an undisturbed atmosphere of constant pressure p_1 and variable density $\rho_1(r)$. The shock conditions provide a differential equation which can be solved to determine $r_2(t)$ and $\rho(r)$ explicitly.

J. H. Giese (Aberdeen, Md.)

5002:

Haskind, M. D. On irreversible and non-equilibrium processes of compression and expansion in gas machines. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1957**, no. 9, 76–81. (Russian)

Consider a cylinder of cross section σ and length l in which a moving piston encloses at time t a mass of gas $\rho_0 l\sigma$ in a volume $(l-\xi(t))\sigma$. First assume that for every t the gas is in equilibrium, with pressure $p_e(t)$, density $\rho_e(t)$, temperature $T_e(t)$, and velocity 0. Now seek solutions of the equations of one-dimensional non-steady viscous flow with heat conduction as perturbations $V(x, t)$, $p_e(t)+p^0(x, t)$, $\rho_e(t)+\rho^0(x, t)$, $T_e(t)+T^0(x, t)$ of the equilibrium approximation. Linearize the equations and consider the limiting case of infinite undisturbed speed of sound. Then $V(x, t)=v(t)(x-l)/(\xi(t)-l)$, where $v=d\xi/dt$, and $\partial T^0/\partial x$ can also be found explicitly. For both uniform and harmonic piston motion (1) heat losses through the ends of the cylinder, and (2) viscous dissipation have been calculated and found to be negligible by comparison with the work of compression for the equilibrium approximation. A similar analysis of non-viscous flow without heat conduction shows non-equilibrium effects also to be negligible for customary values of $v(t)$.

J. H. Giese (Aberdeen, Md.)

5003:

Moriguchi, Haruo. Some nonlinear effect in compressible flow. J. Phys. Soc. Japan 13 (1958), 1510–1516.

The 2-dimensional potential flow of a compressible fluid is considered past a cylindrical obstacle whose cross section is obtained from the circle $\zeta=e^{i\theta}$ by the mapping $z=\zeta+a^2/\zeta-\frac{1}{3}c^4/\zeta^3$. The velocity potential is obtained to the second order in M, the free-stream Mach number, so the results are suggestive only, except when M is small. As c^4 increases, with a^2 fixed, the shape of the obstacle changes, so that for $c>c_0(a)$ and M small, the surface-speed has 4 maxima (near $\theta=\pm\frac{1}{4}\pi, \pm\frac{3}{4}\pi$, say) instead of 2 maxima at $\theta=\pm\frac{1}{2}\pi$. However, as M increases (with a, c fixed) this 4-maximum distribution changes to one with 2 maxima, at least in the cases examined.

T. M. Cherry (Melbourne)

5004:

Power, G.; and Smith, P. A modified tangent-gas approximation for two-dimensional steady flow. J. Fluid Mech. 4 (1958), 600–606.

It is well known that the determination of subsonic gas flows by the hodograph method is greatly simplified if the true relation between pressure p and specific volume v is replaced by a linear approximation $p=a-b^2v$. The present paper proposes that this approximation should be determined by a least squares fitting of the true pv-relation over the range that is significant for the particular problem, and examines (on this scheme) the streaming

flow, for which $M_\infty=0.4$, past a cylinder which is practically circular. The results agree excellently with those obtained, for virtually the same case, by P. E. Lush and T. M. Cherry [Quart. J. Mech. Appl. Math. 9 (1956), 6–21; MR 17, 913] by a variational method. There is, however, a marked discrepancy with what is obtained by the von Karman-Tsien linear approximation (in which $b^2=-(dp/dv)_\infty$), and the suggestion is that this approximation is not so good. *T. M. Cherry* (Melbourne)

5005:

Čuškin, P. I. Subsonic flow of a gas around ellipses and ellipsoids. Vyčisl. Mat. 2 (1957), 20–44. (Russian)

The author considers irrational isentropic subsonic flows around an ellipse (planar flow) and around an ellipsoid of revolution (axially symmetric flow). A method ascribed to A. A. Dorodnicyn is applied to replace the partial differential equations by a finite number of ordinary differential equations which can be solved numerically; boundary conditions on the ellipse or ellipsoid and at infinity can also be satisfied. Results for particular cases are shown graphically; a comparison with results of other writers shows good agreement.

W. Kaplan (Ann Arbor, Mich.)

5006:

Barancev, R. G. On the exact calculation of the supersonic part of a flat nozzle. Vestnik Leningrad. Univ. 12 (1957), no. 13, 89–92. (Russian. English summary)

The exact calculation of the supersonic part of a flat nozzle, when velocity values are given along initial characteristics and nozzle axis, rests on the solution of a Goursat problem.

In this note a method is given for obtaining the exact solution of this problem by reduction to the problem with data given along a characteristic and along free surfaces which was investigated by the author in Dokl. Akad. Nauk SSSR 114 (1957), 955–958 [MR 19, 865]. (From the author's summary)

C. D. Calsoyas (Livermore, Calif.)

5007:

Chang, C. T. A note on the reflection of sound waves at an oblique shock. J. Aero. Sci. 25 (1958), 70–71.

Based on some previous work of the author [J. Aero. Sci. 24 (1957), 675–682; MR 20 #2162] an expression is derived for the reflection coefficient of a sound wave reflected from an oblique shock. Numerical results are presented for a particular case.

P. Chiarulli (Chicago, Ill.)

5008:

Ray, G. Deb. An exact solution of a spherical blast wave under terrestrial conditions. Proc. Nat. Inst. Sci. India. Part A 24 (1958), 106–112.

The equations of spherically symmetrical non-steady flow have solutions for which velocity, pressure, and density are of the form $u=U(\eta)r/t$, $p=P(\eta)/r^3$, and $\rho=\Omega(\eta)t^2/r^5$, where $\eta=tr^{\mu-1}$, $0<\mu<1$. In terms of a single quadrature the author constructs such u, p, ρ that satisfy the shock conditions at a spherical shock moving with constant velocity. The energy within the shock is non-constant, but varies slowly with time for small μ.

J. H. Giese (Aberdeen, Md.)

5009:

MacCamy, R. C. On Babinet's principle. Canad. J. Math. 10 (1958), 632–640.

The principle of Babinet in acoustic and electromagnetic diffraction theory states the equivalences of an aperture S in a plane screen and a plane obstacle occupying the position of S. The author discusses generalizations of this principle, pointing out that both the boundary conditions and partial differential equations may be changed, but that the obstacle must be plane. The emphasis is on the integral equation formulation of boundary value problems. Two examples are discussed.

A. E. Heins (Pittsburgh, Pa.)

5010:

Ting, Lu. Diffraction of disturbances around a convex right corner with applications in acoustics and wing-body interference. J. Aero. Sci. 24 (1957), 821–830, 844.

The author considers the often treated problem of the diffraction of a small disturbance or arbitrary pulse around a 90 degree corner. Through the use of source distributions for the wave equation, the problem is reduced to an integral equation for the pressure distribution along the line of the corner extended. The equation is solved by a successive approximation procedure which converges at least as well as the geometric series of argument $\frac{1}{3}$. Convergence is improved by a regrouping of successive terms so that certain boundary conditions are exactly satisfied for each approximation rather than for the limit.

Three fundamental incident pulses are considered. These correspond to the first three terms of an expansion of a disturbance originating at a point on the wall which is not the corner. The first term corresponds to the classic case of a plane weak shock diffracted by a right corner. The pressure distribution obtained from the second successive approximation is, when plotted, indistinguishable from the exact solution. The other two cases show similar rapidity of convergence. Superposition of these solutions is used to find the diffraction of a weak shock by a rectangular barrier, i.e., more than one corner.

The same solutions are interpreted in terms of analogous problems in linearized three-dimensional steady supersonic flow (the wave equation). In particular, the problem of the interference of a prismatic body of rectangular cross section with a planar wing is treated.

P. Chiarulli (Chicago, Ill.)

5011:

Baron, M. L. Response of nonlinearly supported spherical boundaries to shock waves. J. Appl. Mech. 24 (1957), 501–505.

The author determines the effect of a linearly decaying plane acoustic wave on the boundary of an elastically supported spherical cavity in an infinite acoustic medium. A Laplace transform technique is used to solve two basic problems for which the relation between the medium pressure at the cavity boundary and the corresponding displacement is linear. It is assumed that the zero mode of the motion is adequate to determine the response. These solutions are then used as influence coefficients to develop a non-linear integral equation for the displacement response when the pressure-displacement relation is non-linear. This equation is solved numerically for a given plane wave and pressure-displacement relation. Curves are given for the pressure-time and displacement-time response. *P. Chiarulli* (Chicago, Ill.)

5012:

McVittie, G. C. Some exact solutions of the equations of magnetohydrodynamics when both self-attraction and magnetic fields are present. Rev. Mod. Phys. 30 (1958), 1080–1082.

5013:
✱Cowling, T. G. Magnetohydrodynamics. Interscience Tracts on Physics and Astronomy, No. 4. Interscience Publishers, Inc., New York; Interscience Publishers, Ltd., London; 1957. viii+115 pp. Paperbound $1.75; clothbound $3.50.

This tract gives a concise account of the subject, with particular reference to applications in astronomy and geophysics. The more recent applications to the problem of nuclear fusion are not considered. For the most part the author is concerned with an electrically conducting medium regarded as a continuous incompressible fluid. This excludes certain problems of ionized gases, which are in the province of the preceding volume in this series [L. Spitzer, Jr., Physics of fully ionized gases, Interscience Publ., New York-London; 1956].

In the first chapter the fundamental electromagnetic and mechanical properties of a magnetohydrodynamic medium are established, bringing out the analogy between magnetic diffusivity and kinematic viscosity; e.g., when the "magnetic Reynolds number" is small the lines of force are "frozen" into the material and Walén's analogue of the Helmholtz vorticity equation applies. Laboratory experiments confirming the theory are described. Chapter 2 treats the magnetohydrostatic problems of sunspot equilibrium and solar streamers and filaments. The stability of static configurations is discussed with the aid of Lundquist's energy method. It concludes with a brief account of Chandrasekhar and Fermi's theory of spiral arms. Chapter 3 provides a simplified account of magnetohydrodynamic waves (including the effects of compressibility) and a description of laboratory experiments. The torsional theory and Alfvén's theory of sunspots, and the magnetic oscillation theory of magnetic variable stars are critically discussed. Chapter 4 is concerned with the inhibiting effect of a magnetic field on the two kinds of instability which lead respectively to turbulent and convective flow. Theory and experiment are compared. In chapter 5 dynamo theories of the magnetic fields of bodies such as the earth and the stars are critically described, particularly the work of Bullard and Elsasser. It ends with a valuable discussion of the unresolved question of equipartition between the magnetic and turbulent kinetic energy of a fluid. The last chapter treats the specific case of an ionized gas, paying particular attention to the various forms in which the Ohm's-law relation between the conduction current and the electric fields, real and equivalent, can be expressed. It is shown that in a partially ionized gas the effect of collisions of ions with neutral atoms can become an important mechanism for the dissipation of magnetic energy. The results for fully and partially ionized gases are applied respectively to hot and cold interstellar clouds.

In a field where so much is beyond the reach of experimental verification in the laboratory, this book is particularly valuable for the penetrating physical discussion given throughout the text.

K. C. Westfold (Pasadena, Calif.)

OPTICS, ELECTROMAGNETIC THEORY, CIRCUITS

5014:
Stettler, R. Zur Bestimmung der Wellenflächen gegebener Strahlenbündel. Optik **15** (1958), 407–413.

The author computes the meridional normal trajectories and the caustic for a set of meridional rays coming from an object point, thus obtaining the intersection of the wave surface with the meridional plane and its equation. *M. Herzberger* (Rochester, N.Y.)

5015:
Menzel, E. Die Darstellung verschiedener Phasenkontrast-Verfahren in der optischen Übertragungstheorie. Optik **15** (1958), 460–470.

The author defines the contrast transfer function of an optical system and adds to it a function which determines the phase contrast for self-luminous and partially coherent illuminated objects. The aim is to investigate the effect of phase contrast methods in interference microscope and for use of Zernikes phase contrast methods. The author constructs an object, which he calls a variable sinusoidal phase grating, by means of interference fringes of equal thickness. *M. Herzberger* (Rochester, N.Y.)

5016:
Canals-Frau, D.; et Rosseau, M. Influence de l'éclairage partiellement cohérent sur la formation des images de quelques objets étendus opaques. Opt. Acta **5** (1958), 15–27.

The authors investigate the image of a group of objects (an opaque disc, an opaque line, a half plane and others) in coherent and partially coherent light illuminated with monochromatic light and in the absence of aberration. *M. Herzberger* (Rochester, N.Y.)

5017:
Chako, Nicholas. Application de la méthode de la phase stationnaire dans la théorie de la diffraction des images optiques. C. R. Acad. Sci. Paris **247** (1958), 580–582.

The author investigates a double integral which plays an important role in the theory of diffraction and phase contrast

$$U(k)=\iint_p g(xy)\exp\{ik\Psi(xy)\}dxdy,$$

investigating its critical points, in order to obtain an asymptotic development of U.

The critical points are points in the interior where $\Psi_x=\Psi_y=0$ and the points on the border for which Ψ_s vanishes (s=arc of length of the boundary). To this should be added the points where the boundary has a discontinuous tangent. The author gives asymptotic developments in the two final cases.

M. Herzberger (Rochester, N.Y.)

5018:
Chako, Nicholas. Calcul d'intégrales doubles pour de grandes valeurs d'un paramètre. C. R. Acad. Sci. Paris **247** (1958), 637–639.

The author investigates the integral discussed in the preceding note [#5017], in adding to the critical points described above those which come from an asymptotic value of $g(x, y)$. Under restricting assumptions, he obtains an asymptotic development of $U(k)$ for an interior critical point of the domain.

M. Herzberger (Rochester, N.Y.)

5019:
Amer, S. Non-linear theory of plasma oscillations and waves. J. Electronics Control **5** (1958), 105–113.

The mathematical problem is that of studying the solutions of the nonlinear differential equation $u''-u'^2/(u+a)+b^2u(1+u/a)=0$. Fortunately, this possesses a

first integral of the form $u'=g(u)$, from which many of the desired properties can be deduced.

R. Bellman (Santa Monica, Calif.)

5020:

Wyld, H. W., Jr. Dynamic stability of a self-pinched discharge. J. Appl. Phys. 29 (1958), 1460–1465.

The present paper is an extension of work by Kruskal and Schwarzschild [Proc. Roy. Soc. London Ser. A 223 (1954), 348–360; MR 15, 914] and by M. Rosenbluth [Los Alamos Report LA-2030 (1956)]. These authors investigated the stability of a pinched plasma under static conditions using hydrodynamic methods. The present paper investigates stability in the case of rapid contraction of the plasma. Three different models are used to describe this dynamic plasma: the free particle model, the snowplow model, and the shock wave model. For the first and the third models instabilities can be exhibited easily. In the snowplow model the plasma is stable first, although the author thinks that instabilities probably develop later. The instability times are of the order of the pinch time itself, and a plausibility argument is given as to why this should be. The first two models are discussed in some detail, while the shock wave model, because of analytical difficulties, is considered only in the limiting case when the shock is strong enough to be divorced from the motion of the magnetic piston.

M. J. Moravcsik (Livermore, Calif.)

5021:

Cerkovnikov, Yu. O. On the stability of a plasma. Dopovidi Akad. Nauk Ukraïn. RSR 1957, 461–465. (Ukrainian. Russian and English summaries)

The author investigates the stability of a heterogeneous plasma in a powerful magnetic field. The density of the plasma, its temperature and the magnetic field are regarded as given functions of the coordinates and considered as the parameters on which the plasma stability depends.

Author's summary

5022:

Meyers, Norman H. A Poynting theorem for moving bodies and the relativistic mechanics for extended objects. J. Franklin Inst. 266 (1958), 439–464.

The author adopts a system of electromagnetic equations and a definition of electromagnetic force in a moving material medium proposed by Lan J. Chu and derives from these assumptions an identity having the form of an equation of energy balance analogous to Poynting's identity. Chu's equations are not derived and have not been published elsewhere with supporting argument. Since they are not of a traditional form, this reviewer experienced some difficulty in accepting the results based on them. In treating the relativistic mechanics of continuous media, the author notes that the time dependence of the internal energy leads to a time dependent rest mass, an effect not encountered in relativistic particle mechanics.

R. A. Toupin (Washington, D.C.)

5023:

Iwata, Giiti. Separable dynamical systems of Staeckel in flat space. Progr. Theoret. Phys. 19 (1958), 369–374.

"To help a study in orbits of a charged particle of high energy in electromagnetic fields, we determine coordinate systems and respective potential functions where the Hamilton-Jacobi equation of the particle is integrable by separation of variables." [This paper is an appendix to same Progr. 15 (1956), 513–522; MR 19, 709.]

Author's summary

5024:

Byhovskiĭ, È. B. Solution of the mixed problem for Maxwell's equations in the case of an ideal conducting boundary. Vestnik Leningrad. Univ. 12 (1957), no. 13, 50–66. (Russian. English summary)

The mixed problem for Maxwell's equations over a region Ω with an ideal conducting boundary S has the form

$$(29)\quad \begin{cases} \varepsilon(X,t)\dfrac{\partial E}{\partial t}-\operatorname{rot} H+\sigma_1(X,t)E=F_1(X,t),\\ \mu(X,t)\dfrac{\partial H}{\partial t}+\operatorname{rot} E+\sigma_2(X,t)H=F_2(X,t),\end{cases}$$

$$(30)\quad E|_{t=0}=E_0(X),\ H|_{t=0}=H_0(X),$$

$$(31)\quad E_\tau|_S=0,$$

where $X=(x_1,x_2,x_3)$, E, H, F_1, F_2 are 3-dimensional vectors and E_τ is the tangential component of E relative to S. The author reduces this to Cauchy's problem for the operator equation

$$(1)\quad \mathbf{A}\frac{du}{dt}+S_1u+\mathbf{B}u=f(t).$$

Here $u(t)$ is a vector function belonging to the Hilbert space $\mathscr{H}=L_2(\Omega)\times L_2(\Omega)$ and the coefficients are linear transformations on $\mathscr{H}$ satisfying $S_1=-S_1^*$, $\mathbf{A}=\mathbf{A}^*$, $(\mathbf{A}u,u)\geqq\alpha\|u\|^2$ $(\alpha>0)$, $\|\mathbf{B}u\|\leqq M\|u\|$. Using (1) it is shown that (under suitable conditions on Ω, ε, μ, σ_1, σ_2, F_1, F_2, E_0, H_0) the problem (29), (30), (31) has a weak solution, with $E(X,t)$ and $H(X,t)$ in $L_2(\Omega\times[0,l])$.

M. G. Arsove (Seattle, Wash.)

5025:

Rose, M. E. The electrostatic interaction of two arbitrary charge distributions. J. Math. Phys. 37 (1958), 215–222.

The electrostatic energy of interaction between two separate distributions of charge is obtained in terms of an expansion in the multipole moments of the two distributions. Comparison of the result with a more familiar form of expansion also leads to a new addition theorem for spherical harmonics.

E. T. Kornhauser (Providence, R.I.)

5026:

Wait, James R. Mixed path ground wave propagation. I. Short distances. J. Res. Nat. Bur. Standards 57 (1956), 1–15.

The author considers the problem of the propagation of an electromagnetic wave over a flat plane made up in part of one dielectric and in part of another. The boundary between the dielectrics is a straight line not necessarily at right angles to the line between the transmitting and receiving antennas. An integral equation is formulated and solved by numerical methods for long and medium long wavelengths. The results are compared with experiments and agreement is noted. Finally, the author refines his method of approximation of the integrals in order to permit him to examine oblique incidence of the wave on the boundary. It is found that little refraction occurs at the boundary for such incidence. This last result is admittedly based on arguments having their roots in physical optics.

W. K. Saunders (Washington, D.C.)

5027:

Wait, James R.; and Householder, James. Mixed-path ground-wave propagation. II. Larger distances. J. Res. Nat. Bur. Standards 59 (1957), 19–26.

The results in the paper reviewed above are generalized

to the spherical earth. The work is limited to frequencies below the broadcast band.

W. K. Saunders (Washington, D.C.)

5028:

Wait, J. R. On the mode theory of V. L. F. ionospheric propagation. Geofis. Pura Appl. **37** (1957), 103–115.

The space between the earth and the ionosphere is considered as a wave-guide with sharply bounded walls. Employing a representation in terms of spherical wave functions of complex order, the field of a vertical dipole source is calculated for very low frequencies. It is shown that the zero order mode is heavily damped while higher order modes, especially the first and second, propagate. This is in opposition to the usual assumption. Good agreement is obtained with the experimental results of J. Heritage. *W. K. Saunders* (Washington, D.C.)

5029:

Wait, James R. Propagation of very-low-frequency pulses to great distances. J. Res. Nat. Bur. Standards **61** (1958), 187–203.

The author treats from a waveguide viewpoint the problem of propagation of very-low-frequency pulses radiated by a vertical electric dipole to great distances around the earth. The earth is regarded as a homogeneous, lossy sphere, the ionosphere as a sharply defined reflecting boundary, and the region between these two concentric spheres constitutes the waveguide in which propagation is assumed to take place along an angular direction away from the source. Theoretical results for this idealized model are obtained for both quasi-monochromatic pulses which have a frequency spectrum contained in a narrow band about a central frequency and delta function pulses which have a very broad frequency spectrum. For the quasi-monochromatic pulse, graphical results are presented to show the influence of the ground conductivity and the effective conductivity of the ionosphere on the shape of the envelope of the pulse. Some similar calculations are performed for the impulsive source and compared with experimental data.

L. B. Felsen (Brooklyn, N.Y.)

5030:

Bellman, Richard; and Kalaba, Robert. Invariant imbedding, wave propagation, and the WKB approximation. Proc. Nat. Acad. Sci. U.S.A. **44** (1958), 317–319.

This paper summarizes earlier work on a general principle according to which physical phenomena are described in terms of a sequence of local processes. The application to wave propagation through a stratified medium depends on the reflections against the boundaries of the successive strata that represent such a medium for a stratum thickness tending to zero. The theory leads to a Riccati equation for a quantity $u(z)$ which represents the reflection coefficient of the special part $x>z$ of the inhomogeneous medium. In order to define $u(z)$ we replace the half space $x<z$ by a homogeneous medium having the same properties as do occur actually at $x=z$; $u(z)$ then refers to the reflection at the boundary $x=z$ for a primary wave arriving from $x=-\infty$. The theory comprises the effect of single and multiple reflections associated with the above mentioned fictitious boundaries of the various strata. The characterization of the W.K.B. approximation by the neglect of higher-order reflections is pointed out. This latter approximation is also discussed, with the aid of matrix calculus, for systems of linear differential equations of the second order.

The paper does not contain the details of the underlying computations. An erratum dealing with the incorrect representation of the formulas (5-1) and (5-2) appears in same Proc. **44** (1958), 978.

H. Bremmer (Eindhoven)

5031:

Fedorov, F. I. Inhomogeneous waves and total reflection. Akad. Nauk BSSR Trudy Inst. Fiz. Mat. **1956**, no. 1, 11–31. (Russian)

The author considers the general theory of inhomogeneous waves in an isotropic dielectric and applies the results to the theory of total reflection. Inhomogeneous waves are defined as damped plane waves, which differ from homogeneous waves in that planes of equal phase or equal amplitude are not parallel. Some of the results of this paper were given briefly before [F. I. Fedorov, Dokl. Akad. Nauk SSSR **102** (1955), 69–71; **105** (1955), 465–468; MR **17**, 325; **18**, 169]. Among the conclusions reached in this general vector treatment is the following: In the general case total reflection is accompanied by a specific lateral light pressure, whose force is directed perpendicular to the plane of incidence. *J. E. Rosenthal* (Passaic, N.J.)

5032:

Hochstadt, Harry. Asymptotic formulas for diffraction by parabolic surfaces. Comm. Pure Appl. Math. **10** (1957), 311–329.

Same as Div. Electromag. Res., Inst. Math. Sci., N.Y.U. Res. Rep. No. **EM-89** (1956) [MR **19**, 492].

F. Oberhettinger (Madison, Wis.)

5033:

Keller, Joseph B. Diffraction by an aperture. I. Div. Electromag. Res., Inst. Math. Sci., New York Univ., Res. Rep. No. **EM-92** (1956), ii+61 pp.

The diffraction of waves by an aperture of any shape in a thin screen is treated by a method developed and called by the author "the geometrical theory of diffraction". This is based upon the introduction of new rays — called diffracted rays — which are produced when an incident ray hits the edge of the aperture. This method leads to a formula for the field diffracted by any aperture and the field in the aperture or the far field diffraction pattern can be found. An application to the determination of the far field produced by the incidence of a cylindrical or spherical wave upon a perfectly reflecting half plane shows agreement with the classical results. Explicit formulas and their numerical evaluation are given for the case of slits and circular apertures.

F. Oberhettinger (Madison, Wis.)

5034:

Keller, Joseph B.; Lewis, Robert M.; and Seckler, Bernard D. Diffraction by an aperture. II. Div. Electromag. Res., Inst. Math. Sci., New York Univ., Res. Rep. No. **EM-96** (1956), i+36 pp.

The geometrical theory of diffraction as applied to the diffraction of waves by an aperture of any shape in a thin screen expresses the field u at any point as the sum of three partial fields (u_1, u_2, u_3) termed by the first author as the geometrical optic field, the edge diffracted field and the corner diffracted field, respectively [#5033 above]. The results of this theory are now compared with other approximative methods which express the solution of the problem of the diffraction of waves by an aperture in a thin screen by means of a double integral extended over the aperture and which are known as the Kirchhoff method, the two customary modification of this method and Braunbeck's modification of it. In each case a double integral over the aperture is evaluated asymp-

totically for large k. Each of the three Kirchhoff integrals leads to a sum of three parts u_1', u_2', u_3' in which $u_1'=u_1$. The form of u_2' and u_3' is the same as that of u_2 and u_3, respectively, but the diffraction coefficients of u_2' and u_3' differ from those of u_2, u_3 of the geometrical optic theory. Braunbeck's integral leads to a sum of two fields u_1'' and u_2'' which are exactly identical with u_1 and u_2, respectively. But the Braunbeck method does not apply to corner diffraction. *F. Oberhettinger* (Madison, Wis.)

5035:

Seckler, Bernard D.; and Keller, Joseph B. Diffraction in inhomogeneous media. Div. Electromag. Res., Inst. Math. Sci., New York Univ., Res. Rep. No. **MME-7** (1957), ii+68 pp.

In this report the method developed by Keller [Proc. Symposia Appl. Math. vol. 8, pp. 27–52, Amer. Math. Soc., McGraw-Hill, New York-Toronto-London, 1958; MR **20** #840] for solving problems of diffraction by convex smooth bodies, based on the extended form of Fermat's principle and the introduction of geometrical diffracted rays, has been generalized to cases where the medium is non-homogeneous. The principal modifications of the original method are: (a) The phases of the rays now must be taken as equivalent to the optical paths from the source to the observation point; (b) in the calculation of the amplitude of the field derived from the principle of conservation of energy, one must now consider the variation of the refractive index of the medium. This introduces, not only in the phase, but also in the amplitude of the field, a variation of the refractive index along the total path of the rays, especially the variation of the index in a narrow layer in the obstacle and the medium outside. From the above considerations the authors have derived an expression for the field valid for any point in the lit or shadow region. To illustrate their method and the validity of solutions to diffraction problems gotten by this procedure, the authors have investigated several special types of non-homogeneous media whose index of refraction depends on a single variable (stratified media) with plane and circular diffracting boundaries, for several types of excitation fields for which the corresponding exact or asymptotic solutions of the scalar wave equation can be found. The following problems are treated in this report: (a) Plane wave incidence in a stratified medium along one direction (x-axis), with the refractive index tending to constant values at $x=\pm\infty$; b) a line source in a stratified medium, as above, bounded to one side by a plane; c) a point source with a plane boundary; d) a line source in a cylindrically stratified medium with an imbedded circular boundary; and finally, e) a source at infinity lying between two parallel planes which are caustics. This case gives rise to trapped modes in a plane duct.

In the first part, the analysis is carried out on the basis of the geometrical diffraction theory. The field in both the lit and the shadow region is calculated for all the cases. One interesting feature which does not occur when the medium is homogeneous is that at certain positions (values of x) the direction of the rays becomes vertical and the incident rays turn toward the source. These points are called turning points. The region beyond the turning point is the refracting shadow where no real rays pass. One calls them imaginary or virtual rays and they are obtained by analytical continuation of the real rays. If two turning points exist, then in front of the first turning point one has reflected rays, behind the second point real transmitted rays, and in between virtual rays which give rise to virtual and reflected or transmitted rays when reaching the plane boundaries at the first or the second turning point. For the other cases the shadow boundary separating the reflected region from the refracted shadow or diffracted region is given by the ray which is tangential to the diffracting boundary. In the second part these boundary value problems have been solved from the corresponding scalar wave equation. The asymptotic expressions for the field have been calculated and are in agreement with the expressions derived in part one from the geometrical theory of diffracted rays.

N. Chako (Flushing, N.Y.)

5036:

Oberhettinger, F. On the diffraction and reflection of waves and pulses by wedges and corners. J. Res. Nat. Bur. Standards **61** (1958), 343–365.

"Various problems arising in the theory of the excitation of a perfectly conducting wedge or corner by a plane, cylindrical or spherical wave, are dealt with. The incident wave is represented by a line source (acoustic or electromagnetic) parallel to the edge. The spherical wave is emitted by an acoustic point source or by a Hertz dipole with its axis parallel to the edge. The case of an incident plane wave field is obtained as the limiting case (for large distances of the source from the edge) of the cylindrical or spherical wave excitation." (Author's summary) *A. E. Heins* (Pittsburgh, Pa.)

CLASSICAL THERMODYNAMICS, HEAT TRANSFER

See also 4981, 4991.

5037:

Reismann, Herbert. Two-dimensional periodic flow of heat in polar coordinates. J. Franklin Inst. **266** (1958), 293–300.

Let $T(r, t)$ denote the steady periodic temperatures in the unbounded region outside a cylinder $r=a$ when the temperature of the cylinder is $T_0 \cos \omega t$ and $T(\infty, t)=0$. A formula is derived for $T(r, t)$ in terms of Bessel functions ker and kei, and the variation of the amplitude and phase of the periodic temperature T with radial distance r is examined. A convenient method of determining series that represent such periodic temperatures $T(r, \theta, t)$ for other domaisn with circular boundaries is presented.

R. V. Churchill (Ann Arbor, Mich.)

5038:

Preisendorfer, R. W. A mathematical foundation for radiative transfer theory. J. Math. Mech. **6** (1957), 685–730.

The author gives an axiomatic formulation of the mathematical contents of radiative transfer theory. This theory deals with transport of energy in the form of photons, or of matter in the form of neutrons, in a medium which can absorb, emit and scatter photons or neutrons. The basic equation of transfer is an integro-differential equation. The aim of the present paper is to formulate a set of definitions and axioms general enough to cover the many different cases of transfer theory, and complete enough to derive a generalized equation of transfer and to show under which conditions this equation reduces to the familiar equation. The mathematical methods used throughout are those of Lebesgue measure theory.

L. Van Hove (Utrecht)

5039:

✱Chu, Boa-Teh. Wave propagation in a reacting mixture. 1958 Heat transfer and fluid mechanics institute, held at University of California, Berkeley, Calif., June, 1958: preprints of papers, pp. 80–90. Stanford University Press, Stanford, Calif., 1958. viii+264 pp. $7.50.

In a chemically reacting mixture $p=p(\rho, S, r_1, \cdots, r_{n-1})$, where r_i are the mass fractions of the various species involved. For infinitely fast reactions the r_i assume equilibrium values $r_i=r_i^{(e)}(\rho, S)$, and the equilibrium speed of sound satisfies $a_e^2=\partial p/\partial\rho+\sum(\partial p/\partial r_i)\partial r_i^{(e)}/\partial\rho$. When the reaction rates are finite the characteristic conditions for the partial differential equations of unsteady reacting flow lead to a wave front velocity that satisfies $a^2=\partial p/\partial\rho$. The author shows that as reaction times approach zero the r_i converge to $r_i^{(e)}$ and the coefficients of ∇r_i and $\partial r_i/\partial t$ in the governing partial differential equations approach zero. This singular perturbation behavior is considered to offer the possibility for a non-uniform convergence or "boundary layer" phenomenon which may reconcile the two speeds of sound a_e and a. To clarify this question the author considers a reacting mixture of three species, initially in equilibrium in a cylinder, which is perturbed by piston motion. The equations of one-dimensional non-steady motion are linearized and solved by means of Laplace transforms. For a piston impulsively started to uniform motion the solution reveals a weak wave front that advances with velocity a into undisturbed mixture, and a region of abrupt increase of velocity perturbation that advances with velocity a_e, the abruptness being a decreasing function of reaction time. *J. H. Giese* (Aberdeen, Md.)

QUANTUM MECHANICS

See also 4777, 4982.

5040:

Altshuler, Saul. Variational principles for the wave function in scattering theory. Phys. Rev. (2) **109** (1958), 1830–1836.

"Variational principles are designed for the solution of the Schrödinger equation when a point source is placed in the presence of an inhomogeneous, absorbing medium represented by an arbitrary complex potential function. When the point source is allowed to recede to infinity, these stationary structures reduce to variational principles for the wave function in the standard scattering problem, namely the outgoing solution to the Schrödinger equation for an incident plane wave. Finally, in the asymptotic region, the well-known bifunctional variational principles for the transition amplitudes arise automatically from the stationary forms for the wave function describing the standard scattering problem. A few examples leading to variationally improved wave functions are discussed." (Author's summary)

M. Cini (Rome)

5041:

Imamura, Tsutomu. Criticism on the assumptions in the formal scattering theory. Progr. Theoret. Phys. **18** (1957), 51–65.

"We derive the conditions which give a well-defined meaning to the mathematical expressions appearing in the current formal scattering theory. These conditions remove some restrictions that are usually imposed on the spectra of H and H_0. By these conditions we can systematically discuss many problems, such as the equivalence between two kinds of S-matrices given by Heisenberg and Dyson respectively, or as the adiabatic theorem in the quantum field theory. The rearrangement collisions can be investigated from this view-point." (Author's summary)

M. Cini (Rome)

5042:

Demkov, Iu. N. Symmetry of the coordinate wave function of a many-electron system. Soviet Physics. JETP 34(7) (1958), 491–492 (714–716 of Russian original).

The total wave function of a many-electron system which satisfies the Pauli principle and is an eigenfunction of S^2 may be constructed from coordinate functions and spin functions by using either group theoretical methods or Fock's method. In both cases certain symmetry conditions must be satisfied. In this paper it is shown that the coordinate function obtained from the first satisfies the Fock symmetry conditions.

C. Froese (Vancouver, B.C.)

5043:

Riazanov, G. V. The Feynman path integral for the Dirac equation. Soviet Physics. JETP 33(6) (1958), 1107–1113.

It is shown that, subject to certain assumptions about the motion of the electron, the Feynman path integral is identical with the propagation function of the Dirac equation. *Author's summary*

5044:

Datzeff, Assène. Sur les conditions de Sommerfeld et la mécanique ondulatoire. C. R. Acad. Sci. Paris **247** (1958), 1565–1568.

5045:

Costa de Beauregard, Olivier. Relation entre la densité de spin d'E. Durand et celle de Dirac. Interprétation physique de la relation entre le tenseur inertial de Tetrode et le produit des courants de Dirac et Gordon. C. R. Acad. Sci. Paris **247** (1958), 1965–1967.

5046:

Morand, Max. Sur les fondements géométriques de la théorie des spineurs de l'espace à trois dimensions. C. R. Acad. Sci. Paris **247** (1958), 2299–2301.

5047:

✱Боголюбов, Н. Н.; и Ширков, Д. В. Введение в теорию квантованных полей. [Bogolyubov, N. N.; and Širkov, D. V. Introduction to quantum field theory.] Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1957. 442 pp. (1 insert) 18.50 rubles.

The book is an introduction to the formalism of the quantum theory of fields. Applications to specific physical problems are described only to illustrate the fundamental ideas of the theory. As in all recent efforts of this kind, the authors have had the choice between a mathematically rigorous exposition which would cover very little of what physicists regard as important or a more formal account. The authors have clearly chosen the latter. Although they make considerable effort to describe the theory in a coherent fashion, they do not permit themselves the luxury of ignoring matters which are physically important merely because they are, as yet, mathematically ill defined. Furthermore, by ignoring most mathematical technicalities they are able to write an informative book

5047

which reads easily. (Example: In Chapter IX on dispersion relations, the authors discuss the (dispersion) relations of the form

$$f(E)=\frac{1}{\pi i}\,P\int_{-\infty}^{\infty}\frac{f(E')}{(E'-E)}\,dE'$$

which arise when one considers f analytic in the upper half E plane and vanishing at infinity as fast as $1/|E|$. It is obvious that to insure such a relation one must impose growth conditions on f, not only at infinity, but also for E approaching the real axis. To give a satisfactory discussion of these conditions would require the technical theory of distributions and would not greatly contribute to the readers' understanding of the main ideas of the theory, so the authors ignore the matter.) The book is divided into nine chapters ranging from the most elementary matters (Chapter I, The classical theory of free fields; Chapter II, The quantum theory of free fields) to those at the frontier of existing knowledge (Ch. VII, The method of functional integration; Ch. VIII, The renormalization group; and Ch. IX, Dispersion relations.) The intervening chapters constitute a systematic account of the advances in the theory of fields mainly associated with the names of Schwinger, Feynmann and Dyson. (Ch. III, The scattering matrix; Ch. IV, The removal of divergences from the scattering matrix; Ch. V, Application of the general theory of the removal of divergences to special cases; Ch. VI, The Schroedinger equation and dynamical variables.) Ch. I contains mainly standard material explained in a clear and economical way: the action principle, transformation law of fields under inhomogeneous Lorentz transformation, Noether's theorem relating the invariance of the Lagrangian to the existence of conservation laws, its application to derive the form of the energy momentum and angular momentum, an explicit discussion of the cases of a scalar field neutral and charged, a charged vector field, the electromagnetic field, and a charged spin $\frac{1}{2}$ field, a compact introduction to Dirac's equation, the electromagnetic interaction, the Fermi interaction, gauge invariance of the first and second kind. Ch. II gives the quantum field theory of free fields of the types discussed in Ch. I. The assumptions that the energy be non-negative, that the fields transform as they ought under Lorentz transformations and that the energy momentum operator be the same function of the fields (up to ambiguities of order and a constant term) as in the classical theory are shown to lead uniquely to the standard forms of the theories. An original and gratifying feature is the discussion of the ordered products of free field operators. By using the regularization method of Pauli and Villars, the authors establish that all coefficients in normal products expansions can be given a well defined meaning as distributions. Ch. III defines the collision matrix and evaluates it within the framework of perturbation theory. The exposition follows the ideas of Stueckelberg, Rivier, Green and the authors. It is assumed that the theory under discussion can be imbedded in a family of theories in which the interaction is multiplied by a smooth function $g(x)$ (which can provide adiabatic switching on and off of the interaction). The collision matrix then becomes a function $S(g)$ of $g(x)$. The requirements of relativistic invariance, local commutativity, etc. are formulated in terms of $S(g)$ and its functional derivatives with respect to $g(x)$. It is then shown that if $S(g)$ is assumed to be of the form

$$1+\sum_{n=1}^{\infty}\int\cdots\int S_n(x_1\cdots x_n)g(x_1)\cdots g(x_n)dx_1\cdots dx_n,$$

then the $S_n(x_1\cdots x_n)$, $n>1$, are essentially uniquely determined in terms of $S_1(x_1)$. (Actually, there is in each order an arbitrariness which is closely connected with a certain arbitrariness in the definition of a time-ordered product of field operators.) The general theory is applied to give the S-matrix in normal form and the Feynman rules for its calculation. The information required to make practical calculations of cross sections is explained and illustrated in the case of second order Compton scattering, pair annihilation, and Bremstrahlung. Ch. IV begins with a discussion of the divergences of quantum electrodynamics in lowest order. The expressions for S_2 and S_3 are shown to be well defined when regularized, and the removal of the parts which become infinite in the limit when the regularization parameters approach infinity is discussed. This procedure is extended to the general case, following work by Bogolyubov and Parasiuk; it is a significant improvement of previous treatments of the removal of divergences. The general theory leads to the classification of $S(g)$ as of the first or second kind, depending on whether a finite or infinite number of arbitrary constants have to be fixed in defining it. Ch. V discusses the details of the removal of divergences for the theory of a neutral scalar field ϕ with self coupling $\lambda\phi^3(x)$, for spinor electrodynamic and for pseudoscalar meson theory. The results of this analysis are then applied to derive the well-known equation of Schwinger for the complete Green's functions in the case of electrodynamics. The exposition is based on the expansion in $g(x)$, but has the great advantage of being entirely free from infinities and ambiguity for suitably chosen g. Chapter VI beats its way back from $S(g)$ to the Schroedinger equation in the form given it by Tomonaga and Schwinger, obtains expressions for the infinitesimal operators of the Lorentz group and other dynamical quantities in terms of the coupled fields and applies the resulting formalism to some external field problems: vacuum polarization and Lauch shift. Chapter VII begins by establishing the connection between certain vacuum expectation values of time ordered products of field operators and (purely formal) integrals over function space. These results are then used to obtain closed formulae for the complete Green's functions of pseudoscalar meson theory in terms of functional integrals. The analogous results for electrodynamics are then used to determine the effect on the electron and photon Green functions of a gauge transformation. Finally, the method is used to obtain an exact expression for the Green's function in the model of Bloch and Nordsieck. The chapter succeeds in making the method of functional integration alluring, while warning the reader that it has provided no essentially new insight into the structure of the theory. Ch. VIII is devoted to the exploitation of an invariance property which the complete Green's functions and vertex part possess but which, however, is not possessed by the partial sums of the corresponding perturbation series: their transformation law under the multiplicative renormalization group. This property is first used to derive identities connecting the momentum dependence and charge dependence of the Green's functions. These identities are then used to develop estimates for the ultraviolet and infra-red behavior of the Green's functions and vertex part. Analogous functional equations are obtained for the case of psuedo-scalar meson theory. Ch. IX is an introduction to the theory of dispersion relations to which the first-named author made such a decisive contribution. A brief historical introduction is

followed by a section in which the fundamental requirements on the structure of the theory used in Ch. III to study the perturbation series are reformulated in terms independent of perturbation theory. The authors choose to express everything in terms of functional derivatives with respect to the incoming field. {In the reviewer's opinion, the introduction of functional derivatives is an irrelevant complication; the whole business is neater and certainly just as rigorous when expressed in terms of the field operators in the Heisenberg picture.} The standard Källén-Lehmann spectral representation for the Green's functions are derived. Then the scattering amplitude for mesons on nucleons is expressed in terms of advanced and retarded matrix elements and the proof of the dispersion relations for forward scattering outlined. The analytic continuation of the amplitude in the case of non-forward scattering is then discussed. The argument is based on a difficult theorem on analytic continuation whose proof is given elsewhere. The explicit isotopic spin structure of the scattering amplitude for pion-nucleon scattering is then disentangled and the dispersion relations obtained. The book closes with a few remarks on the future of the relativistic field theory. The authors take the view that the internal consistency of quantum electrodynamics remains an open problem, and that experimental tests of dispersion relations are important to see whether nature indicates the existence of a fundamental length. {In the opinion of the reviewer, the book is an original and very valuable contribution to the literature of quantum field theory. It by no means settles the fundamental problems of the subject (in fact, it is somewhat characteristic that even the problems which it solves with rigor are generally not stated with rigor), but it is an excellent jumping-off place for the student.} *A. S. Wightman* (Princeton, N.J.)

5048:

✱Costa de Beauregard, O. Théorie synthétique de la relativité restreinte et des quanta. Les grands problèmes des sciences, VIII. Préface de M. Lévy. Gauthier-Villars, Paris, 1957. xii+200 pp.

This is an exposition of the relativistically covariant formulation of quantum mechanics and quantum field theory. It includes some of the author's work, in addition to that of others. The main subjects treated are the Minkowski-covariant form of the theory of complementarity, relativistic analytical mechanics of a point charge, wave mechanics of a particle without and with spin, the formalism of von Neumann, superquantization, the Tomonaga-Schwinger quantum theory of fields, the Feynman rules and the Dyson method of successive approximations, the Wheeler and Feynman treatment of electrodynamics, and the problem of the microscopic symmetry and macroscopic asymmetry between past and future. The author not only presents the mathematical formalism, but also discusses physical concepts and interpretations.

N. Rosen (Haifa)

5049:

Gotō, Ken-iti. Quantization of non-linear fields. Nuovo Cimento (10) 3 (1956), 533–550.

"A method of developing quantized non-linear field theories is proposed in the functional representation, where the non-linearity of the classical equation appears as the peculiar form of field operators and the field equation for state vectors becomes a linear functional differential equation. The method to diagonalize the field Hamiltonian is illustrated by the free electromagnetic field, the hydrodynamical field and the non-linear meson field." (Author's summary) *M. Cini* (Rome)

5050:

Sommerfield, Charles M. The magnetic moment of the electron. Ann. Physics 5 (1958), 26–57.

The magnetic moment of the electron can be expressed as a power series in the fine structure constant α. The zeroth order term of this series is the well known $eh/4\pi mc$ of the Dirac theory. The linear term in α (the so-called second order radiative correction) was calculated by Schwinger [Phys. Rev. (2) 73 (1948), 416–417; 76 (1949), 790–817; MR 11, 569]. The quadratic term (the fourth order radiative correction) was first computed by Karplus and Kroll [Phys. Rev. (2) 77 (1950), 536–549] by the Feynman-Dyson technique for constructing matrix elements for observable quantities. The present author has also calculated this fourth order correction. The results of the calculation were published elsewhere [Phys. Rev. (2) 107 (1957), 328–329], and now in the present paper the details of this calculation are given. The technique used is that of the mass-operator formalism of Schwinger [Proc. Nat. Acad. Sci. U.S.A. 37 (1951), 452–455, 455–459; MR 13, 520], which makes the calculation much simpler than that of Karplus and Kroll. This technique is illustrated by the calculation of the second order moment, and then the calculation of the fourth order moment is sketched. The result for the fourth order moment disagrees with the result of Karplus and Kroll, although the sign of the contribution is the same in both calculations.

M. J. Moravcsik (Livermore, Calif.)

5051:

Baumann, Kurt. Retardierte Produkte und Bindungszustände. Z. Physik 152 (1958), 448–453.

The author investigates the properties of products of field operators where the operators are retarded with respect to an arbitrary number of points $z_1, \cdots, z_k$. These products appear when one expands the ordinary product $\psi(z_1)\cdots\psi(z_k)$ in incoming fields. The vacuum expectation value of these generalized retarded products can, in perturbation theory, be characterized with the aid of a generalization of the Dyson double graphs. (From the author's summary) *G. Källén* (Lund)

5052:

Drell, S. D.; and Zachariasen, F. Form factors in quantum electrodynamics. Phys. Rev. (2) 111 (1958), 1727–1735.

The techniques of dispersion relations are applied here to the study of the electromagnetic form factors of the electron in pure quantum electrodynamics. It is assumed that the electromagnetic vertex function at infinite momentum transfer vanishes sufficiently rapidly for the dispersion relation which it satisfies to contain only convergent integrals. (This so-called "no-subtraction" philosophy has previously been applied with success to pion-nucleon scattering and photoproduction [Chew, Goldberger, Low and Nambu, same Rev. (2) 106 (1957), 1337–1344, 1345–1355; MR 19, 920, 1019].) To solve the resulting coupled integral equations, the approximation is made of retaining from the expansion in intermediate states only that which contains one extra $e\bar{e}$ pair and no extra photons. It turns out that the solution thus obtained cannot be made to reproduce perturbation theory at low momentum transfers and is therefore inconsistent with experiment. However, it is not clear whether this

indicates a failure of the no-subtraction philosophy or the inadequacy of the approximations. *P. W. Higgs* (London)

5053:

Bollini, C. G. Quantization of zero mass fields. I. Tensor fields. Repub. Argentina. Publ. Com. Nac. Energia Atomica. Ser. Fis. **1** (1957), 261–274. (Spanish)

A symmetric tensor of rank s which satisfies the field equations

$$\partial_\nu\partial_\nu A_{\nu_1\cdots\nu_s}=0, \tag{1}$$

$$\partial_{\nu_1} A_{\nu_1\cdots\nu_s}=0, \tag{2}$$

$$\partial_{\nu_1\nu_2} A_{\nu_1\cdots\nu_s}=0 \tag{3}$$

has $2s+1$ independent components and, on quantization, is associated with bosons of spin s and mass zero. In the usual method of quantization all the components of A (subject only to the requirement of symmetry) are treated as independent dynamic variables satisfying the wave equation (1), from which the usual covariant commutation relations follow. It then turns out that the remaining equations are inconsistent with the commutation properties of the field operators and must be imposed as subsidiary conditions on the state vector: thereby one eliminates all but two polarization states of the quanta.

The author develops an alternative procedure which is a generalization to higher spins of that due to J. G. Valatin [Danske Vid. Selsk. Mat.-Fys. Medd. **26** (1951), no. 13; MR **13**, 805] in the case $s=1$. Equations (1), (2) show that A is perpendicular to a null vector ∂_ν. It may therefore be resolved into components parallel to ∂_ν or perpendicular to the plane defined by ∂_ν and a constant timelike unit vector n_ν. In this way one obtains $s+1$ tensors $A^{(r)}{}_{\nu_1\cdots\nu_s}$ $(r=0, \cdots, s)$, where $A^{(r)}$ is r-fold transverse, $(s-r)$-fold longitudinal. When equation (3) is taken into account, each $A^{(r)}$ $(r\neq 0)$ may be expressed in terms of the two independent components of a traceless symmetric rth rank tensor in the 2-space normal to ∂_ν and n_ν, i.e., in terms of two rth rank polarization tensors. $A^{(0)}$ has just one component. Thus one obtains $2s+1$ "canonical coordinates of the field", which may be quantized in the usual way. Under gauge transformations only the two components of $A^{(s)}$, the totally transverse tensor, are invariant: it is these which are associated with the spin s particle. The commutation relations which are now satisfied by $A_{\nu_1\cdots\nu_s}$ are consistent with all the field equations, but depend explicitly on n_ν. *P. W. Higgs* (London)

5054:

Ohmae, Akira; and Senba, Kei. On the integral forms of the covariant Fock equations of the field theory. Mem. Fac. Engrg. Hiroshima Univ. **1** (1957), 9–16.

A new method is described for deriving the equations originally given by Matthews and Salam [Proc. Roy. Soc. London Ser. A **221** (1953), 128–134; MR **15**, 586].

J. C. Taylor (London)

5055:

Kirshnits, D. A. On a functional relation in quantum mechanics. Soviet Physics. JETP **34**(7) (1958), 717–719 (1037–1039 of Russian original).

It is proved that for a system of non-interacting particles in a stationary state in an external field $V(\mathbf{r})$ the number density $\rho(\mathbf{r})$ which is a function of V and all its derivatives satisfies the following equation:

$$D\rho/DV=\partial\rho/\partial V,$$

where D/DV is the Euler derivative. This relation is claimed to be useful to derive and check approximate expressions for the density. *D. ter Haar* (Oxford)

5056:

Ulehla, I. A new possible theory of the μ-meson. Nuovo Cimento (10) **9** (1958), 679–693.

A twenty-component spinor equation is proposed for the mu-meson; it differs from the Dirac equation by terms involving a "parameter of anomaly" g. The electromagnetic coupling of this field is studied. To first order in g, the effect is to add a Pauli anomalous moment term to the Dirac equation. This implies that the magnetic moment of the mu-meson is $1+(2\pi)^{-1}\alpha-2g$ (including lowest order radiative corrections). A further effect of a Pauli term is to introduce a quadratic divergence into the lowest order mass-renormalization, again proportional to g. It is then suggested that the high mu to e mass ratio and the deviation of the observed mu-moment from $1+(2\pi)^{-1}\alpha$ can be obtained by fixing g and the value of the cutoff (taking the lowest order self-mass seriously), and that the anomalous high-energy mu-scattering can also be accounted for with the same g.

S. Deser (Waltham, Mass.)

5057:

Mandelstam, S. Determination of the pion-nucleon scattering amplitude from dispersion relations and unitarity. General theory. Phys. Rev. (2) **112** (1958), 1344–1360.

This paper contains the first discussion in the literature of a new representation of scattering amplitudes in relativistic dispersion relations which has given a new impetus to theoretical physics of elementary particles. The case discussed in the paper is that of pion-nucleon scattering, but the results are really more general. The main idea behind the new representation is that it exhibits the analytic properties of the scattering amplitude as a function of the energy and momentum transfer simultaneously. On account of the inherent symmetry of the representation, pion-nucleon scattering must be discussed together with pion pair production by nucleon-antinucleon pair. Thus, both the pion-nucleon and the pion-pion coupling constants appear in the formalism. The assumptions underlying the new representation have not been proven rigorously, but at least they can be justified in perturbation theory. In practice, the calculations are made neglecting states with more than two particles, and using a method of iteration which is analogous to that used in the static meson theory. It is hoped that the new method will avoid formal difficulties, such as the existence of ghost states which are suspected to result from the neglect of crossing symmetry. The paper gives a detailed treatment of the application to pion-nucleon scattering, including the discussion of the unitarity condition and the subtraction terms. It is found that in the one-meson approximation the unitarity condition cannot be satisfied at all energies if crossing symmetry and the analytic properties are to be maintained. For sufficiently low values of the coupling constant, however, it is still possible to obtain a unique procedure.

M. J. Moravcsik (Livermore, Calif.)

5058:

Feld, Bernard T. Mesons and the structure of nucleons. II. The nucleon isobar and pion dynamics. Ann. Physics **4** (1958), 189–232.

The "atomic model" used in an earlier paper [same Ann. **1** (1957), 58–76; MR **20** #3728] for the study of the physical nucleon states is here extended to the higher energy region by postulating the existence of a nucleon

"isobar" state of spin 3/2 and isotopic spin 3/2. The dynamical properties of the nucleon, like the scattering and photoproduction of pions and Compton scattering on nucleons, are computed in terms of this model. The contribution to these various processes from the isobar resonance and from alternative channels are explicitly evaluated and compared with available experiments. Certain speculations on the mechanism of the flat "resonance" in the neighborhood of 1 B.e.v. in terms of the excitation of the isobar are presented. This paper, incidentally, provides an excellent introduction to pion dynamics. *E. C. G. Sudarshan* (Cambridge, Mass.)

5059:

Hartogh, C. D.; and Tolhoek H. A. Cluster developments for Jastrow wave functions. I. Physica 24 (1958), 721–741.

For the ground state of a Fermi gas with short range interactions an approximate wave function can be obtained by variational methods. The trial function Φ suggested by Jastrow [Phys. Rev. (2) 98 (1955), 1479–1484] is a Slater determinant for the unperturbed ground state wave function, Φ_0, multiplied by a correlation factor, F, which is a symmetric function of the particle coordinates and a function of the variational parameters. In evaluating the energy integral and other expectation values it is useful to define

$$g_k(r_k, r_k')=\int \Phi^*(r_k', r_{N-k})\Phi(r_k, r_{N-k})dr_{N-k},$$

where r_k denotes the coordinates of a set of k particles. This paper deals with the cluster expansion of g_k for a large system of spinless fermions where $F=\prod f(r_{ik})$, the product extending over all pairs of particles within the system (r_{ik} is the distance between particles i and k).

This paper gives the technique, while subsequent papers will deal with applications. The formal machinery is analogous to the standard methods used in statistical mechanics, except for the important difference that in this case the statistical correlations, i.e., correlations due to Fermi statistics only, are not restricted to configurations in which the particles are within the range of their dynamical interactions. This, then, requires a different grouping of terms in the expansion to enforce convergence.

The results are as follows: g_k can be expanded in a series; the series probably converges if the average number of particles within the range of dynamical interaction, δ, is small, i.e., if $n\delta^3 \ll 1$, n being the number-density; the successive terms in the series depend in two different ways on the density: they contain increasing powers of $n\delta^3$ and they are functions of the Fermi momentum of the system; to obtain an explicit expression for the terms in the series a sufficient condition is derived on the $f(r)$ function in terms of which F is defined; this expansion can also be used for a boson gas by putting $\Phi_0 \equiv 1$ (in this case the cluster expansion reduces to an expansion in density, since the dependence on the Fermi momentum is absent). *N. L. Balazs* (Chicago, Ill.)

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See also 5022, 5048.

5060:

Koniukov, M. V.; and Terletskii, Ia. P. Relativistic motion of an electron in an axially symmetric field which moves along the axis of symmetry. Soviet Physics. JETP 34(7) (1958), 692–693 (1003–1005 of Russian original).

5061:

Romer, Robert H. Twin paradox in special relativity. Amer. J. Phys. 27 (1959), 131–135.

The familiar "traveling twin paradox" is discussed in a particularly simple manner, using special relativity only. The asymmetrical aging of the twins is predicted using only the relativistic definition of simultaneity and the relativistic time dilatation. Possible objections to such a treatment are discussed. *Author's summary*

5062:

Builder, G. Ether and relativity. Austral. J. Phys. 2 (1958), 279–297.

The author argues that the relative retardation of clocks in special relativity demands our recognition of the existence of absolute velocities. He supplements this argument by an ancillary discussion of the electrodynamical interaction of two point particles. He also argues that the identification of inertial frames depends on the absolute character of acceleration. He rejects the hypothesis that the observed behaviour of bodies can be explained by the distribution of matter in the universe. Instead he advocates the ether hypothesis, according to which absolute space plays a causal role in physical phenomena. *G. J. Whitrow* (London)

5063:

Vogtherr, Karl. Die Massenveraenderlichkeit nach der Relativitaetstheorie. Methodos 9 (1957), 183–207. (Text in German and English)

5064:

Pham, Mau Quan. Sur le principe de Fermat. Enseignement Math. (2) 4 (1958), 41–70.

The author derives Fermat's principle for the geodesics in a general relativistic medium. Starting from the relativistic derivation of Maxwell's equations and the geodesics of zero length on the corresponding Riemann geometry the author introduces a variation principle, which leads to a specific form of Fermat's principle; then he describes the basic laws of electromagnetic radiation in a general relativistic field.

M. Herzberger (Rochester, N.Y.)

5065:

Kompaneets, A. S. Strong gravitational waves in free space. Soviet Physics. JETP 34(7) (1958), 659–660 (953–955 of Russian original).

The author considers a line element of the form

$$-ds^2=Adx_1^2+Cdx_2^2+2Bdx_2dx_3+Ddx_3^2-Adx_4^2,$$

where A, B, C and D depend only on x_1 and x_4. A solution of the field equations in free space corresponding to two non-linear interacting cylindrical waves is obtained. The waves are propagated with the fundamental velocity. It is also pointed out that shock waves are not necessarily formed. *G. L. Clark* (London)

5066:

Yilmaz, Huseyin. New approach to general relativity. Phys. Rev. (2) 111 (1958), 1417–1426.

An ingenious attempt is made in this paper to discuss the theory of gravitation as a covariant scalar field theory. In the new theory the principle of equivalence remains preserved and Einstein's field equations are still valid, but the interpretation of the stress-energy tensor $T_\mu{}^\nu$ is different. In Einstein's theory it is the stress-energy of matter alone, and the gravitational field is not included

in it. In the present theory it is the stress-energy tensor of the gravitational field that is employed, and Einstein's equations

$$8\pi T_\mu{}^\nu = R_\mu{}^\nu - \tfrac{1}{2} g_\mu{}^\nu R,$$

are algebraic identities. The actual field equation is the wave-equation

$$\phi^\mu{}_{;\mu} = -4\pi(-g)^{-\frac{1}{2}} \Sigma_J M_g{}^J \delta(x-x^J),$$

where $M_g{}^J$ are the strengths of the gravitating mass singularities at the points x^J, $\delta(x-x^J)$ is the δ function and the semicolon denotes covariant differentiation. Whereas in Einstein's theory the law of gravitation in a matter-free part of space is $R_{\mu\nu}=0$, in the present theory the corresponding law is $R_{\mu\nu}=2\phi_\mu\phi_\nu$, $\phi^\mu{}_{;\mu}=0$. A functional solution $g_{\mu\nu}(\phi)$ for a static set of singularities is shown to be (1) $ds^2 = \exp(-2\phi)dt^2 - \exp(2\phi)(dx^2+dy^2+dz^2)$, and in the case of a single mass singularity $\phi = M_g/r$. Although the two interpretations are different from each other, the solutions of the equations predict the same numerical results with regard to the three crucial tests of general relativity. A further feature of the present theory is that the self-energy of a point singularity does not diverge but is equal to $M_g c^2$.

A new theory of cosmology is also presented. If the mass density has the same value everywhere, the theory requires a line-element representing a steady-state universe. The applications of the theory to the time-scale problem and to the derivation of the Mach principle are discussed.

The author also points out that the line-element (1) is still valid if ϕ is changed by an additive constant and suggests that possibly the constant is not arbitrary but its value is such that any local observer will observe light local to himself to propagate with the same velocity in all directions regardless of his position in the gravitational field. *G. L. Clark* (London)

5067:

✱Hlavatý, Václav. Geometry of Einstein's unified field theory. P. Noordhoff Ltd., Groningen, 1957. xxxii+341 pp. $9.00.

Since the first appearance of a proposed new unified field theory by Einstein, the author has written a series of fifteen to twenty papers dealing with the geometrical and physical aspects of this theory. The present book is a culmination of these papers. Since most of this work has already been adequately reviewed we shall confine ourselves to remarks of a more or less general nature.

Einstein based his theory upon a real non-symmetric tensor g_{ij} and implied that the equations

$$(1) \qquad \frac{\partial g_{ij}}{\partial x^k} = \Gamma_{ik}{}^a g_{aj} + \Gamma_{kj}{}^a g_{ia}$$

would determine the components of the linear connection $\Gamma_{jk}{}^i$. A large part of the present book is devoted to determining the necessary and sufficient conditions for a solution of (1) to exist and to obtaining explicit forms of the solution in tensorial form. The remaining portion of the book is devoted mainly to geometric and physical aspects of the complete theory.

From a mathematical point of view the book leaves little to be desired. Certain problems are enunciated and solutions are given with the clarity, and detail that is characteristic of the many papers that the author has written in this field. From a physical point of view, the reviewer feels that the book falls far short of the desired goal.

We quote from the preface. "Let us start with problem (2) and denote by $f_{\lambda\mu}$ a tensor field which could be identified with that of the electromagnetic field. Such a field must satisfy Maxwell's equations. Moreover, these equations must appear as a consequence of the purely geometrical conditions (I.15), since Maxwell's equations must constitute part of the unified theory." The author then goes on to obtain a unique $f_{\lambda\mu}$ based on the above condition and the requirement that $f_{\lambda\mu}$ can be obtained from g_{ij} by a tensorial construction. The reviewer agrees that in this way the author has avoided the problem of the physical identification of the field tensors that has bothered many other people working with this theory. The reviewer does not agree with the basis by means of which this unique identification has been obtained. In building a unified field theory it would seem sufficient to have Maxwell's equations appear as approximations of the field equations and not as an exact part of these equations. In fact, having a distinct electromagnetic field satisfying distinct equations of its own seems, in a large measure, to destroy the claim of such a theory to be classified as a unified field theory.

It is regrettable that the author has confined his book so closely to a description of his own research. The inclusion, discussion and comparison of the work and ideas of other people would have enhanced the present book. Professor Hlavatý has greatly extended the tensorial techniques one might use in the investigation of unified field theories. To this extent the present book is a welcome addition to the literature. *M. Wyman* (Edmonton, Alta.)

5068:

Stephenson, G. La géométrie de Finsler et les théories du champ unifié. Ann. Inst. H. Poincaré **15** (1957), 205–215.

The paper consists of three parts. In the first part, the author points out the difference between Einstein-Maxwell field equations of general relativity and Einstein's latest unified field equations. The first set yields the equations of motion in the form

$$(1) \qquad \frac{d^2x^\nu}{ds^2} + \left\{ {\nu \atop \lambda\ \mu} \right\} \frac{dx^\lambda}{ds}\frac{dx^\mu}{ds} + F_\mu{}^\nu \frac{dx^\mu}{ds} = 0$$

($F_\mu^\nu dx^\mu/ds$=Lorentz' force vector). Callaway's application of the EIH method to the second set does not yield any Lorentz force and therefore the motion of a charged particle and of an uncharged particle would be the same.

In the second part, the author discusses the attempt to describe unified field theory by means of a Finsler metric

$$ds = A_\lambda dx^\lambda + \sqrt{(g_{\lambda\mu} dx^\lambda dx^\mu)},$$

which leads by means of $\delta\int ds=0$ to (1) with $F_{\mu\lambda} = 2\partial_{[\lambda}A_{\mu]}$. However the tensor $R_{\mu\lambda}$ does not yield any appropriate scalar term which could be taken as Lagrangean.

In the third part, the author suggests that one could build a theory based on the description of nuclear forces by means of a scalar and presents two illustrations based either on

$$\frac{d}{d\tau}\left(m\frac{dx^i}{d\tau}\right) = -\frac{\partial V}{\partial x^i},$$

or on

$$\Box^2 V - \mu^2 V = 0$$

in classical relativistic theory.

{Remarks of the reviewer: 1) If $F_{\mu\lambda}$ is of the second or third class (which is always the case in Callaway's approximation) we could have $F_{\mu}{}^{\nu}(dx^{\mu}/ds)=0$ in (1) for $F_{\mu\lambda}\neq 0$. 2) The clue to Callaway's result is that four Einstein's equations $\partial_{[\omega}R_{\mu\lambda]}=0$ do not contribute anything to the equations of motion of the considered singularities. If one replaces these four differential equations of the third order by another set of four differential equations of the third order, then the kinematical description of the motion results in the form

$$(2)\qquad \frac{d^2x^\nu}{dq^2}+\Gamma_{\lambda}{}^{\nu}{}_{\mu}\frac{dx^\lambda}{dq}\frac{dx^\mu}{dq}=0.$$

There is such a coordinate system for which the first approximation of (2) is the classical Newton gravitational law and the second approximation acquires the form (1). [See pp. 192, 212 of #5067 above.] 3) The Einstein unified theory contains an explicitly given scalar function $\mathscr{C}$ which plays an important role in the construction of (2) [loc. cit., pp. 45, 184].} *V. Hlavatý* (Bloomington, Ind.)

5069:

Bonnor, William Bowen. Les équations du mouvement en théorie unitaire d'Einstein-Schrödinger. I. Ann. Inst. H. Poincaré **15** (1957), 133–145.

In the first two sections the author discusses the gravitational theory of Einstein and the Einstein-Maxwell theory from the point of view of the equations of motion (of singularities) which one obtains by EIH method. In the third section, the author first mentions Callaway's objection to the latest Einstein unified field theory (see reviewer's remark 2 in the preceding review) and then proposes a slightly different set of the 14 unified field equations

$$(1)\quad {}^*R_{(\mu\lambda)}=0,\ \partial_{[\omega}{}^*R_{\mu\lambda]}=0,\ [{}^*R_{\mu\lambda}\overset{\text{def}}{=}R_{\mu\lambda}+p^2U_{\mu\lambda}],$$

where

$$U_{\mu\lambda}\overset{\text{def}}{=}g_{[\lambda\mu]}-g^{[\alpha\beta]}g_{\mu\alpha}g_{\beta\lambda}+\tfrac{1}{2}g^{[\alpha\beta]}g_{\beta\alpha}g_{\mu\lambda},\quad p=\text{const}.$$

Applying Callaway's method to this system he obtains up to the fourth approximation

$$(2)\qquad \overset{1}{m}\frac{d^2\mathfrak{r}}{dt^2}=\frac{\mathfrak{r}}{r^3}(-\overset{1}{m}\overset{2}{m}+p^2q^2\overset{1}{e}\overset{2}{e}),\quad q=\text{const}.$$

The author states that he was unable to find an exact solution of (1) even for a third class.

{Reviewer's remarks: 1) The author's information (p. 135) that the reviewer rejected Einstein's 82 equations for the unified field theory is misleading. Only four of these 82 equations have been replaced by another set of four differential equations. 2) The author's statement (p. 143) that all solutions of Einstein's unified field equations are singular at the origin is wrong. [See pp. 180, 186 of #5067 above.] 3) If one has to accept (1) as a basis for a unified field theory, then it must be shown that these equations admit an exact solution at least for a plane electromagnetic wave. 4) Despite the title of the present paper the equations (2) are not equations of motion in the presence of the electromagnetic tensor field, because Lorentz's force vector is missing.}

V. Hlavatý (Bloomington, Ind.)

5070:

Bonnor, William Bowen. Les ondes gravitationnelles en relativité générale. II. Ann. Inst. H. Poincaré **15** (1957), 146–157.

Work on plane and cylindrical waves is reviewed. A solution free from singularities, already published by the author [J. Math. Mech. **6** (1957), 203–214; MR **19**, 228], is included. The work of I. Robinson, not yet published, is quoted concerning an exact solution of the field equations for the case of plane waves.

[Editor's note: This paper and the one reviewed above are the first two in a sequence; the third was listed in MR **19**, 1023].

N. Rosen (Haifa)

5071a:

Ikeda, Mineo; and Abe, Shingo. On tensorial concomitants of a non-symmetric tensor $g_{\mu\nu}$. I. Tensor (N.S.) 7 (1957), 59–69.

5071b:

Ikeda, Mineo. On tensorial concomitants of a non-symmetric tensor $g_{\mu\nu}$. II. Tensor (N.S.) 7 (1957), 117–127.

Let $g_{\lambda\mu}\neq g_{\mu\lambda}$ be the basic tensor of Einstein's unified field theory, $h_{\lambda\mu}\overset{\text{def}}{=}g_{(\lambda\mu)}$, $k_{\lambda\mu}\overset{\text{def}}{=}g_{[\lambda\mu]}$, $k_{\lambda}{}^{\nu}\overset{\text{def}}{=}k_{\lambda\mu}h^{\lambda\nu}$, ${}^{(2)}k_{\lambda}{}^{\nu}\overset{\text{def}}{=}k_{\lambda}{}^{\alpha}k_{\alpha}{}^{\nu}$, ${}^{(p)}k_{\lambda}{}^{\nu}={}^{(p-1)}k_{\lambda}{}^{\alpha}k_{\alpha}{}^{\nu}$. Denote by $\mathscr{C}(T_1, T_2, \cdots)$ tensorial concomitant of the tensors $T_1, T_2, \cdots$. The first paper is concerned with $C\,(g_{\lambda\mu})$ for the first two classes. The second paper deals with $C(g_{\lambda\mu}), \mathfrak{E}^{\omega\mu\lambda\nu}, \mathfrak{e}_{\omega\mu\lambda\nu})$ ($\mathfrak{E}$, $\mathfrak{e}$ are indicators). The investigation is based on the reviewer's canonical coordinate system [J. Rational Mech. Anal. **1** (1952), 539–562; MR **14**, 416]. A typical statement of the first paper: "$\delta_{\lambda}{}^{\nu}$, ${}^{(p)}k_{\lambda}{}^{\nu}$, $p=1, 2, 3$, form a complete set of $C(g_{\lambda\mu})$ of mixed valence two; for the second class ${}^{(3)}k_{\lambda}{}^{\nu}$ has to be excluded." Typical theorem about $C(g_{\mu\lambda}, \mathfrak{e}_{\omega\mu\lambda\nu})$: For any odd m,

$${}^{(m+4)}k_{\lambda\nu}\mathfrak{h}-2\mathfrak{h}K\,{}^{(m+2)}k^{\omega\mu}\mathfrak{e}_{\omega\mu\lambda\nu}+\mathfrak{k}\,{}^{(m)}\tilde{k}_{\lambda\nu}=0.$$

Here, $\mathfrak{h}$, $\mathfrak{k}$, $\mathfrak{g}$ are determinants of $h_{\lambda\mu}$, $k_{\lambda\mu}$, $g_{\lambda\mu}$, ${}^{(p)}\tilde{k}_{\lambda\nu}\overset{\text{def}}{=}{}^{(p)}k^{\omega\mu}\mathfrak{e}_{\omega\mu\lambda\nu}\sqrt{|\mathfrak{h}|}/2$, $p=1, 3, 5, \cdots$, and $2K\mathfrak{h}=\mathfrak{g}-\mathfrak{h}-\mathfrak{k}$. {Reviewer's remark: The majority of results could be obtained more easily from the formula ${}^{(p)}k_{\lambda}{}^{\nu}=\sum_x(\underset{x}{\lambda})^p\overset{x}{a}_{\lambda}\underset{x}{a}^{\nu}$, where $\underset{x}{\lambda}$, $\underset{x}{a}^{\nu}$, $\overset{x}{a}_{\lambda}$ are the eigenvalues and the corresponding eigenvectors of $k_{\lambda}{}^{\nu}$ [Hlavatý, #5067above].}

V. Hlavatý (Bloomington, Ind.)

5072:

Wyman, Max; and Zassenhaus, Hans. Zero curvature tensor in Einstein's unified field theory. Phys. Rev. (2) **110** (1958), 228–236.

[The numbers in brackets refer to the pages and equations of the book #5067 above.]

Einstein's latest unified field theory is based on the equations (1a) $D_{\omega}g_{\lambda\mu}=2S_{\omega\mu}{}^{\alpha}g_{\lambda\alpha}$, (1b) $S_{\lambda\alpha}{}^{\alpha}=0$, (1c) $R_{\mu\lambda}=\partial_{[\mu}X_{\lambda]}$ ($R_{\mu\lambda}\overset{\text{def}}{=}R_{\alpha\mu\lambda}{}^{\alpha}$, X_λ arbitrary) [132; (2,7), (2,8)]. Here D_{ω} is the symbol of the covariant derivative with respect to the unified connection $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$; $S_{\lambda\mu}{}^{\nu}$ and $R_{\omega\mu\lambda}{}^{\nu}$ are the torsion and curvature tensors of $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$.

The authors replace (1c) by the stronger condition (2a) $R_{\omega\mu\lambda}{}^{\nu}=0$, so that the second algebraic identity concerning $R_{\omega\mu\lambda}{}^{\nu}$ reduces to (2b) $D_{[\omega}S_{\mu\lambda]}{}^{\nu}+2S_{[\omega\mu}{}^{\alpha}S_{\lambda]\alpha}{}^{\nu}=0$ [129; (1.5)b], while the integrability conditions of (1a) reduce by virtue of (2a) to (2c) $D_{[\lambda}S_{\omega]\mu}{}^{\nu}+3S_{[\omega\mu}{}^{\alpha}S_{\lambda]\alpha}{}^{\nu}=0$ [130; (1.6)b]. The equations (2b), (2c) are compatible if and only if (3a) $D_{\omega}S_{\mu\lambda}{}^{\nu}=0$, (3b) $S_{[\omega\mu}{}^{\alpha}S_{\lambda]\alpha}{}^{\nu}=0$.

The equations (2a), (3a), (3b) are integrability conditions of (4a) $D_{\mu}\overset{i}{a}_{\lambda}=0$, (4b) $D_{\mu}\underset{j}{a}^{\nu}=0$, (4c) $D_{\mu}\overset{i}{b}_{\lambda}=2S_{\mu\lambda}{}^{\alpha}\overset{i}{b}_{\alpha}$, (4d) $D_{\mu}\underset{j}{b}^{\nu}=-2S_{\mu\lambda}{}^{\nu}\underset{j}{b}^{\lambda}$, where $\overset{i}{a}_{\lambda}$, $\overset{i}{b}_{\lambda}$ are two sets of four linearly independent vectors and $\underset{j}{a}^{\nu}$, $\underset{j}{b}^{\nu}$ are their inverse. Hence if $g_{ij}=g_{ji}$ is an arbitrary system of constants of rank 4 then (5) $g_{\lambda\mu}=g_{ij}\overset{i}{a}_{\lambda}\overset{j}{b}_{\mu}$ is a solution of (1a).

If $\Gamma_j{}^i{}_k$ and $S_{jk}{}^i$ are nonholonomic components of $\Gamma_\lambda{}^\nu{}_\mu$ and $S_{\lambda\mu}{}^\nu$ with respect to the frame $\underset{j}{a}{}^\nu$, $\overset{i}{a}_\lambda$, then (6a) $\Gamma_j{}^i{}_k=0$, (6b) $S_{jk}{}^i\overset{j}{a}_\lambda\overset{k}{a}_\mu=\partial_{[\mu}\overset{i}{a}_{\lambda]}$, and consequently the nonholonomic form of (3) reduces to (7) $S_{jk}{}^i=\text{const}$, $S_{[ab}{}^j S_{c]j}{}^d=0$. {The authors derived (4), (6b), (7) by another approach.}

Using the canonical form of the constants $S_{jk}{}^i$ [L. P. Eisenhart, Continuous groups of transformations, Princeton Univ. Press, 1933; p. 155] the authors solved (6b) and obtained from (4a) the $\Gamma_\lambda{}^\nu{}_\mu$ necessary for solving (4c). Substituting the solutions a, b into (5), they obtained the tensor $g_{\lambda\mu}$ satisfying (1a). On the basis of their investigation of solutions of the Einstein formulation of his unified field theory the authors conclude that this theory is incomplete. For the above mentioned solutions the equations of motion seem capable of being appended to the field equations in more than one way.

{Remarks of the reviewer. (i) This objection was to be anticipated. The equations (1b), (1c) lead to a unique skew symmetric tensor $f_{\lambda\mu}$ (an algebraic concomitant of $g_{\lambda\mu}$) for which (1b), (1c) are equivalent to Maxwell equations [167, 169] and to the gravitational field equations and yield the kinematical description of the trajectory of a charged particle [176–195]. The condition (2) demotes (1c) to identities which are the cause of the above mentioned ambiguity. (ii) The same objection holds also for the pure gravitational theory of relativity. In fact, if the holonomy group is degenerate, then there is an infinite set of quadratic tensors of rank 4 (the components of which are not proportional), which have the same Christoffel symbols and the same curvature tensor. Therefore, one could say that a pure "Mathematical approach is doomed to failure", as the authors believe to be the case with the unified field theory.}

V. Hlavatý (Bloomington, Ind.)

5073:

Wrede, Robert C. "n" dimensional considerations of basic priciples A and B of the unified theory of relativity. Tensor (N.S.) 8 (1958), 95–122.

The reviewer solved the Einstein equations in the space-time

$$D_\omega g_{\lambda\mu}=2S_{\omega\mu}{}^\alpha g_{\lambda\alpha} \tag{1}$$

for the unknowns $\Gamma_\lambda{}^\nu{}_\mu$ by means of the natural nonholonomic frame induced by the nonsymmetric tensor $g_{\lambda\mu}$ [#5067 above]. The author generalizes this method for an arbitrary n using some restrictions imposed on the eigenvalues $\underset{1}{\lambda}, \cdots, \underset{2m-w}{\lambda}$ of $k_\lambda{}^\nu$ ($w=0$ for $n=2m$, $w=1$ for $n=2m+1$). He deals with all three cases

$$K_{2m-w}\neq 0,$$
$$\begin{cases}K_{2j-w}\neq 0,\\ K_{2j+2-w}=K_{2j+4-w}=\cdots=K_{2m-w}=0\ (j=1,\cdots,m-1),\end{cases}$$
$$K_{2-w}=K_{4-w}=\cdots=K_{2m-w}=0,$$

where

$$K_{2p-w}\overset{\text{def}}{=}k_{\underset{1}{\lambda}}{}^{\overset{1}{\lambda}}\cdots k_{\underset{p}{\lambda}}{}^{\overset{p}{\lambda}}$$

($w=0$ for n even, $w=1$ for n odd).

To obtain the solution the author investigates in detail these scalars. Applying the results of this investigation to the problem given by (1) the author finds its solution in all three cases (2). The method is too involved to be reproduced in a short review.

V. Hlavatý (Bloomington, Ind.)

5074:

Kerr, R. P. On spherically symmetric solutions in Moffat's unified field theory. Nuovo Cimento (10) 8 (1958), 789–797.

J. Moffat [Proc. Cambridge Philos. Soc. 52 (1956), 623–625; 53 (1957), 473–488; MR 18, 332; 21 #2514] a formé des équations du champ en termes complexes, $R_{\mu\nu}=-\lambda g_{\mu\nu}$, généralisant les équations de la Relativité générale, à l'aide d'un tenseur $g_{\mu\nu}$ complexe symétrique et d'une connexion complexe Γ déduite de ce tenseur ($g_{\mu\nu}$ représente alors à la fois les champs électromagnétique et gravitationnel). L'auteur étudie la solution générale statique à symétrie sphérique en $g_{\mu\nu}$ de ce système d'équations et montre qu'elle dépend de deux fonctions réelles arbitraires, une fois éliminées les indéterminations dûes aux changements de repères. Il pense que cette indétermination — qui n'existe pas physiquement pour un champ donné par les conditions aux frontières — prouve que le système proposé par J. Moffat est incomplet et essaie de le compléter sont par une relation d'équivalence entre les $g_{\mu\nu}$ [analogue à celle déduite de l'invariance de jauge pour le champ électromagnétique], soit par une condition d'isothermie $\mathfrak{G}^{\mu\nu}{}_{,\nu}=0$ ($\mathfrak{G}^{\mu\nu}=\sqrt{(-g)}g^{\mu\nu}$), supposant aussi qu'il doit y avoir une solution unique en $g^{\mu\nu}$, seulement dans le cas de coordonnées isothermes, lesquelles seraient alors les seules coordonnées physiquement admissibles.

L'étude des équations du mouvement dans le cas quasi-statique sera faite ultérieurement. *J. Renaudie* (Rennes)

ASTRONOMY

See also 4475, 5013, 5038.

5075:

Pars, L. A. Inequalities occurring in the restricted problem of three bodies. J. London Math. Soc. 32 (1957), 355–356.

The inequalities concern relative minima of a certain function of one variable; cf. Wintner, The analytical foundations of celestial mechanics, Princeton Univ. Press, 1941 [MR 3, 215], 359–365, whose argument the present author asserts is invalid.

5076:

Kranjc, A. Determinazione di un'orbita circolare ed effemeride mediante calcolatrici elettroniche a programma. Mem. Soc. Astr. Ital. (N.S.) 29 (1958), 217–231. (English summary)

The classical expressions used for the determination of a circular orbit and ephemeris are not satisfying in the case of an electronic computing machine; therefore the resolution of the fundamental equation is made with the method of cords, which has a slow convergence. Then the new modifications are described; the calculation of orbit and ephemeris for thirty successive points with the same time lag is made. The time taken by the calculation has been $2^m10^s+25^s$ at each point of ephemeris.

Changes are proposed for the calculation of the circular orbit of an artificial satellite. *Author's summary*

5077:

Härm, R.; and Schwarzschild, M. Numerical integrations for the stellar interior. Astrophys. J. 1 (1955), Suppl. no. 10, 319–430.

This paper gives tables of solutions of differential

equations which describe approximately parts of the structures of stars. The logarithms of the variables are given; the independent variable is $\log r/r_0$ or $\log(r_0-r)/r$, where r is the distance from the centre and r_0 is the radius of the star. The differential equations are based on simplifying assumptions about the physical laws. It is assumed that the opacity is given by a modification of a law due to Kramer. Nuclear energy generation in the core is assumed to be due to the proton-proton reaction.

All but one of the differential equations were integrated by hand computation, amounting to about a year's work. Several numerical integrations of the remaining equation were performed by the computer at the Institute for Advanced Study at Princeton, N.J., U.S.A. The solutions have not been fitted so as to give the structure of a complete star but may be regarded as a basis for further work. Since the publication of this paper, electronic computers have been programmed to fit solutions [cf. C. B. Haselgrove and F. Hoyle, Monthly Not. Roy. Astr. Soc. 116 (1956), 515–526; MR 18, 939] so that the structure of a model may be derived directly.

C. B. Haselgrove (Manchester)

5078:

Uneo, Sueo [Ueno, Sueo]. La méthode probabiliste pour les problèmes de transfert du rayonnement. La réflexion diffuse et la transmission dans l'atmosphère finie avec la diffusion non cohérente. C. R. Acad. Sci. Paris **247** (1958), 1557–1559.

5079:

Tidman, D. A. Structure of a shock wave in fully ionized hydrogen. Phys. Rev. (2) **111** (1958), 1439–1446.

The shock is treated by the Fokker-Planck equations for the proton and electron distribution functions F and f. The Ansatz is made that F is the sum of two Maxwell distributions referring, respectively, to thermal motion about the mean stream velocity up- and down-stream from the shock front, with different temperatures and spatial densities which are functions of position; the electron function f is approximated by a single Maxwell distribution about a variable streaming velocity with variable space density and temperature. Arguments for the validity of these approximations are given. Relations are derived for the streaming velocities, temperatures, and densities up- and down-stream from the shock in terms of the Mach number. The author also calculates and gives curves for the shock thickness, the distance behind the shock in which the electrons and protons regain thermal equilibrium with one another, and the distance over which the electrons adjust their mean streaming velocity to that of the protons, all as functions of Mach number.

A. Herzenberg (Manchester)

5080:

Prendergast, Kevin H. The equilibrium of a self-gravitating incompressible fluid sphere with a magnetic field. II. Astrophys. J. **128** (1958), 361–374.

In part I [same J. 123 (1956), 498–507; MR 19, 927] the author had obtained an exact solution of the equations of hydromagnetic equilibrium, but had left the questions of uniqueness and stability for future investigation. Here he examines the limits imposed by the conditions (i) that the pressure must be everywhere positive and (ii) that the model must be dynamically stable.

Condition (i) leads directly to the requirement that the magnetic energy should not exceed 0.5069 times the gravitational energy. Condition (ii) is applied by means of a variational principle for the case where (1) the equilibrium configuration is spherical and (2) the magnetic field vanishes on and outside the surface. The calculation of the eigenvalues is difficult; however, in the axisymmetric case it is possible to arrive at an estimate after choosing a simple form for the displacement. This leads to the requirement that the magnetic energy should not exceed 0.4068 times the gravitational energy.

K. C. Westfold (Sydney)

5081:

Chandrasekhar, S.; and Woltjer, L. On force-free magnetic fields. Proc. Nat. Acad. Sci. U.S.A. **44** (1958), 285–289.

In a force-free magnetic field, i.e., $j\times H=0$ where j is current density and H the strength of the field, the field strength satisfies $\nabla\times H=\alpha H$, where α is an arbitrary function of position. In a medium with infinite electrical conductivity a necessary condition for the steady state magnetic energy to be a maximum is that the field be force-free with constant α. The isoperimetric variational problem giving this result can be regarded from the dual point of view to give the corresponding result for the minimum magnetic energy dissipation for a fixed magnetic energy.

R. G. Langebartel (Urbana, Ill.)

GEOPHYSICS

5082:

Fleagle, Robert G. On the mechanism of large-scale vertical motion. J. Meteorol. **15** (1958), 249–258.

This is mainly an investigation of the way in which large scale vertical motions in the atmosphere give rise to the transformation of potential into kinetic energy. Using a linearized theory the author derives an expression for the vertical velocity in terms of measurable parameters. The physical meanings of the various terms in the expression are discussed, and the accuracy of the theory is assessed by comparing calculated vertical velocities with corresponding results using other methods.

M. H. Rogers (Shrivenham)

5083:

Jensen, Eberhart. Toroidal oscillations of an incompressible conductive fluid sphere in a decay field. Astrophys. J. Suppl. Ser. **16** (1955), 141–166.

"The decaying magnetic field in a conducting homogeneous fluid sphere which has the longest decay time is a dipole field outside the sphere. In this initial state it is assumed that no material motions are present. If this state is perturbed, hydromagnetic oscillations about the initial field will result. If periods short compared to the decay time are considered, it is shown how the eigenvalue matrix for the characteristic frequencies can be found in the case of toroidal oscillations. The modes of the velocity field and of the perturbed magnetic field are asymmetrical in the sense that the associated Legendre functions involved have $m>0$. The eigen-value matrix is symmetrical and thus gives only real values of ω^2. Since the matrix is nondiagonal, the exact values of the characteristic frequencies cannot be found. The approximate solution obtained by taking into account only the terms of lowest order in the matrix shows that both positive and negative values of ω^2 appear. In one of the cases considered, corresponding to a rigid boundary, the eigen-values

apparently form a continuous spectrum. An application is made to the terrestrial field." (Author's summary)

K. C. Westfold (Sydney)

OPERATIONS RESEARCH AND ECONOMETRICS

See also 4571, 4713.

5084:

Lieberman, Gerald J. LIFO vs FIFO in inventory depletion management. Management Sci. 5 (1958), 102–105.

Given a set of items to be used for specific tasks, two extreme policies are LIFO (last in, first out), in which the youngest item on hand is always used first, and FIFO (first in, first out) in which the oldest item is used first. The author studies both types of policies, and gives sufficient conditions under which each is optimal.

R. Bellman (Santa Monica, Calif.)

5085:

Ammeter, Hans. Risikotheoretische Grundlagen für die Bestimmung des Maximums des Selbstbehaltes. Verzekerings-Arch. Actuar. Bijv. 35 (1958), 101–106.

In the Lundberg-Cramèr collective risk theory [see, e.g., Harald Cramèr, Jubilee Volume, Försäkring. Skandia, Skandia Insurance Co, Stockholm, 1955; MR 19, 779], the author has introduced changing basic loss probabilities. Here, he refers to his paper in Verzekerings-Arch. 35 (1958), 219–246, giving the necessary formula system and numerical calculations for the distribution of losses, the probability of loss, the security loading and the maximum of self retention.

P. Johansen (Copenhagen)

5086:

Charnes, A.; and Cooper, W. W. Management models and industrial applications of linear programming. Management Sci. 4 (1957), 38–91.

Distinguishing between three areas of mangement activity, viz. planning, operations and control, the authors review the field, with reference to a bibliography of 83 titles.

5087:

Wagner, Harvey M. A practical guide to the dual theorem. Operations Res. 6 (1958), 364–384.

By means of elementary matrix operations the author describes the various variations on the simplex method, with emphasis on the relation between these techniques, the modified simplex method and the dual theorem.

W. Freiberger (Providence, R.I.)

5088:

Charnes, A.; and Cooper, W. W. The theory of search: optimum distribution of search effort. Management Sci. 5 (1958), 44–50.

A discrete analogue of a problem considered by Koopman [Operations Res. 5 (1957), 613–626; MR 19, 819] is treated as a problem of convex programming.

J. Kiefer (Oxford)

5089:

Tyndall, D. G. Welfare pricing and transport costs. Management Sci. 5 (1959), 169–178.

An algorithm is presented for a generalized version of the transportation problem, with supply and demand depending on local prices. The algorithm is iterative and will presumably terminate in a finite number of steps, at least in the simpler cases. No proofs are given, but there are some numerical examples. The author claims that the method will work in cases of decreasing cost, but it seems that in those cases it may come close to a mere enumeration of all possibilities; in the more frequently considered case of constant or increasing cost it appears to be simple enough. There are some disturbing misprints in the mathematical exposition of the problem.

H. S. Houthakker (Cambridge, Mass.)

5090:

Wagner, H. M.; and Whitin, T. M. Dynamic problems in the theory of the firm. Naval Res. Logist. Quart. 5 (1958), 53–74.

The authors consider the problem of maximizing the profits of a firm operating over several periods of time. In each period there is given total cost and total revenue as a function of output; in addition, there is a set-up cost for any period in which production takes place and a unit inventory cost. All costs and revenues may vary with time. Because of the existence of set-up costs, the problem is partly combinatorial in nature. The solution is a recursive algorithm, based essentially on the point that the optimal sales and production decision for any period can be determined easily from the net inventory change in the period. Then for the last period, given desired final inventory, optimal decisions can be stated as a function of incoming inventories. With this information, optimal decisions for the last two periods can be given as a function of inventories coming into the next-to-last period, and so forth. The solution for the period as a whole is finally determined by the initial inventory for the first period.

In the special case where marginal costs are constant and identical in all periods, there is considerable simplification, since a firm will never both produce in and carry inventory into the same period.

K. J. Arrow (Stanford, Calif.)

5091:

Rapoport, Anatol. Critiques of game theory. Behavioral Sci. 4 (1959), 49–66.

BIOLOGY AND SOCIOLOGY

See 4894, 4897.

INFORMATION AND COMMUNICATION THEORY

See also 4895, 4896.

5092:

Shapiro, H. S.; and Slotnick, D. L. On the mathematical theory of error-correcting codes. IBM J. Res. Develop. 3 (1959), 25–34.

Abstract: "Hamming considered the problem of efficient, faultless transmission of binary data over a noisy channel. For a channel which corrupts no more than one binary digit in each sequence of length *n*, he constructed alphabets, the so-called Hamming codes, which permit error-free signalling. The authors study the analogous problem for channels which can corrupt a greater number of digits. Non-binary channels are also studied, and analogues of the Hamming codes are constructed. It is perhaps of interest that some of the techniques employed derive from algebraic and analytic number theory, mathe-

matical disciplines not generally associated with the type of applied problems considered in this paper."

The paper is well written, fairly easy to follow and contains results of interest in its field.

R. W. Hamming (Murray Hill, N.J.)

5093:

Kay, I.; and Silverman, R. A. On the uncertainty relation for real signals. Information and Control 1 (1957), 64–75.

It is generally believed that there exists an "uncertainty relation" of the type $\Delta t \Delta \omega \geqq \frac{1}{2}$ between the effective duration (Δt) of a signal, and its effective bandwith ($\Delta \omega$). It was noted by D. Gabor [J. Inst. Elec. Engrs. Part III. 93 (1946), 429–457] that the usual proof requires a certain modification when the signal is real, and E. Wolf recently showed [Div. Electromag. Res., Inst. Math. Sci., New York Univ., Res. Rep. No. **EM-106** (1957); also Proc. Phys. Soc. **71** (1958), 257–269] that the inequality is not necessarily valid for a real signal with a non-vanishing spectral component of zero frequency. The present paper supplements these considerations. A modified "uncertainty relation" for real signals is established, and it is shown that real signals exist for which the uncertainty product is appreciably less than one-half.

E. Wolf (Manchester)

5094:

Adam, A. Entropie und Streuung. Metrika 1 (1958), 99–110.

The author asserts that information theory has many points of contact with statistics. Indeed, the entropy of communication theory is a fundamental statistical concept of similar far reaching importance as variance. The author supports his assertion with some indications of the structural and conceptual relationship of the entropy measure H and the variance σ^2. These indications cover axiom 3' of D. K. Faddeev [Uspehi Mat. Nauk (N.S.) **11** (1956), no. 1 (67), 227–231; MR **17**, 1098], a formulation of regression analysis, and a comment on a particular property of the normal distribution.

S. Kullback (Washington, D.C.)

5095:

✶Chiang, Tse-Pei. Eine Bemerkung zur Definition der Information. Arbeiten zur Informationstheorie, II, pp. 61–64. Mathematische Forschungsberichte. VI. VEB Deutscher Verlag der Wissenschaften, Berlin, 1958. 77 pp. DM 14.40.

German translation of Teor. Veroyatnost. i Primenen. 3 (1958), 99–103 [MR **20** #789].

5096:

✶Gelfand, I. M.; Kolmogoroff, A. N.; und Jaglom, A. M. Zur allgemeinen Definition der Information. Arbeiten zur Informationstheorie, II, pp. 57–60. Mathematische Forschungsberichte. VI. VEB Deutscher Verlag der Wissenschaften, Berlin, 1958. 77 pp. DM 14.40.

German translation of Dokl. Akad. Nauk SSSR (N.S.) **111** (1956), 745–748 [MR **18**, 859].

5097:

✶Gelfand, I. M.; und Jaglom, A. M. Über die Berechnung der Menge an Information über eine zufällige Funktion, die in einer anderen zufälligen Funktion enthalten ist. Arbeiten zur Informationstheorie, II, pp. 7–56. Mathematische Forschungsberichte. VI. VEB Deutscher Verlag der Wissenschaften, Berlin, 1958. 77 pp. DM 14.40.

German translation of Uspehi Mat. Nauk (N.S.) **12** (1957), no. 1(73), 3–52 [MR **18**, 980].

5098:

✶Rajski, C. The Bayes rule and the entropy. Transactions of the first Prague conference on information theory, statistical decision functions, random processes held at Liblice near Prague from November 28 to 30, 1956, pp. 35–36. Publishing House of the Czechoslovak Academy of Sciences, Prague, 1957. 354 pp. Kčs 34.00.

CONTROL SYSTEMS

5099:

✶Bonamy, M. Servomécanismes. Théorie et technologie. Collection d'ouvrages de mathématiques à l'usage des physiciens. Masson et Cie, Editeurs, Paris, 1957. 284 pp. 4800 fr.

The work is divided into two essentially distinct discussions. The first deals with the general theory of servomechanisms, and the second with their technology. In this second part is a description of the various components of electrical servomechanisms together with a briefer discussion of fluid devices. The first part presents an extended analysis of linear servomechanisms. In this, except for numerous examples, units are disposed of in the first chapter and the discussion has been freed from the specific nature of the components of the systems involved. The amount of mathematical detail, justification, and background varies widely from section to section. For example, a solution of the cubic equation is given at one place while operational methods, which are used extensively, are introduced in a very sketchy manner. Part one concludes with a short chapter introducing non-linear systems. *M. F. Ruchte* (Ames, Iowa)

HISTORY AND BIOGRAPHY

5100:

Taton, René. Réaumur mathématicien. Rev. Hist. Sci. Appl. 11 (1958), 130–133.

Réaumur, in 1708 and 1709, presented three papers on curves to the French Academy, when he was 25–26 years of age. These are his only mathematical papers. A sketch is given of their content (general conchoid, imperfect involutes). *D. J. Struik* (Cambridge, Mass.)

5101:

Thebault, Victor. French geometers of the 19th century. Math. Mag. 32 (1958), 79–82.

5102:

Gonçalves, J. Vicente. Recherches modernes sur les limites des racines des polynomes. Univ. Lisboa. Revista Fac. Ci. A (2) 7(1958), 57–88.

This is an expository article on the unified theory of research on bounds for the roots of polynomial equations. The author begins with the work of Florimond Debeaune, who was the first to write on the idea of bounds. This was followed by the work of Rolle, Newton, and Lagrange. In the nineteenth century, Cauchy considered the roots of the general polynomial equation. His formulas are discussed here, as well as the subsequent work by Hayashi,

Hurwitz, Kakeya, Lucas, Pellet, Walsh, Parodi, Eneström, Karamata, Tomić. Next are given formulas for upper limits which were established in the twentieth century by Carmichael-Mason, Jensen, Birkhoff, Fujiwara, Kuniyeda, Berwald, Kojima, Gonçalves, Walsh, Anghelutza, Westerfield. In all of these formulas only the absolute values of the coefficients appear. Next, formulas are given in which the actual values of the coefficients are used. These were found by Williams, Gonçalves, Wall, E. Frank, Brauer, Parodi. Formulas also are given for variable limits for certain of the roots. These were found by Landau, Fejer, Allerdice, Nagy, Markovitch, Montel, Vythoulskas, Van Vleck, Biernacki, Anghelutza, Gonçalves, Fujiwara, Hayashi, Egervary, Carmichael-Mason, Specht. There is a bibliography of fifty references to the work cited here. *E. Frank* (Chicago, Ill.)

5103:

Nicolescu, Miron. On the mathematical activity of the Mathematical Institute of the Rumanian Academy and of the Chair of Analysis of the University of Bucarest. Mat. Lapok 7 (1956), 18–25. (Hungarian)

5104:

Szénássy, Barna. The mathematical work of I. Martinovics (1755-1795). Mat. Lapok 7 (1956), 277–290. (Hungarian. Russian summary)

5105:

Obláth, Richárd. Gyula Vályi (25. Jan. 1855-13. Oct. 1913). Mat. Lapok 7 (1956), 61–70. (Hungarian)

5106:

Grimshaw, M. E. Hans Ludwig Hamburger. J. London Math. Soc. 33 (1958), 377–383.

A brief biography and a bibliography of 48 items.

5107:

Schmidt, Hermann. Zum Gedächtnis an Fritz Lettenmeyer, † 1953. Jber. Deutsch. Math. Verein. 61 (1958), Abt. 1, 2–6.

A personal and scientific bibliography, with a list of 14 mathematical works by Lettenmeyer.

5108:

Schmidt, Hermann. Hermann Ludwig Schmid †. Jber. Deutsch. Math. Verein. 61 (1958), Abt. 1, 7–11.

A brief biography. For a report of Schmid's scientific work the author refers to Hasse, Math. Nachr. 18 (1958), 1–18 [MR 20#1615].

5109a:

Anonymous. Obituary: Neumann János. Mat. Lapok 8 (1957), 1–7. (Hungarian)

5109b:

Anonymous. Supplement to the list of the works of J. von Neumann. Mat. Lapok 8 (1957), 210.

A brief biography, followed by a bibliography of von Neumann's scientific work: 124 titles, and 11 more in the Supplement.

5110:

Szőkefalvi-Nagy, Béla. J. von Neumann's work in the theory of operators. Mat. Lapok 8 (1957), 185–210. (Hungarian)

5111:

Tarján, Rezső. The work of J. von Neumann concerning electronic computing machines. Mat. Lapok 9 (1958), 6–18. (Hungarian)

5112:

Anonymous. Obituary: Riesz Frigyes. Mat. Lapok 7 (1956), 1–9. (Hungarian)

5113:

Anonymous. The list of scientific works of Ch. Jordan. Mat. Lapok 7 (1956), 291–294.

There are listed 85 papers by Károly Jordan, several of them in non-mathematical sciences.

5114:

Anonymous. The list of works of L. Fejes Tóth, winner of the Kossuth-prize in 1957. Mat. Lapok 8 (1957), 281–283.

Scientific bibliography of 69 items.

5115:

Anonymous. The list of the papers of the late M. Fekete. Mat. Lapok 9 (1958), 1–5.

A scientific bibliography of 79 entries.

5116:

Anonymous. The list of works of P. Erdös, winner of the Kossuth-prize in 1958. Mat. Lapok 9 (1958), 136–147.

274 articles are listed, including some awaiting publication.

5117:

Bosanquet, L. S. Hung Ching Chow. J. London Math. Soc. 33 (1958), 383–384.

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